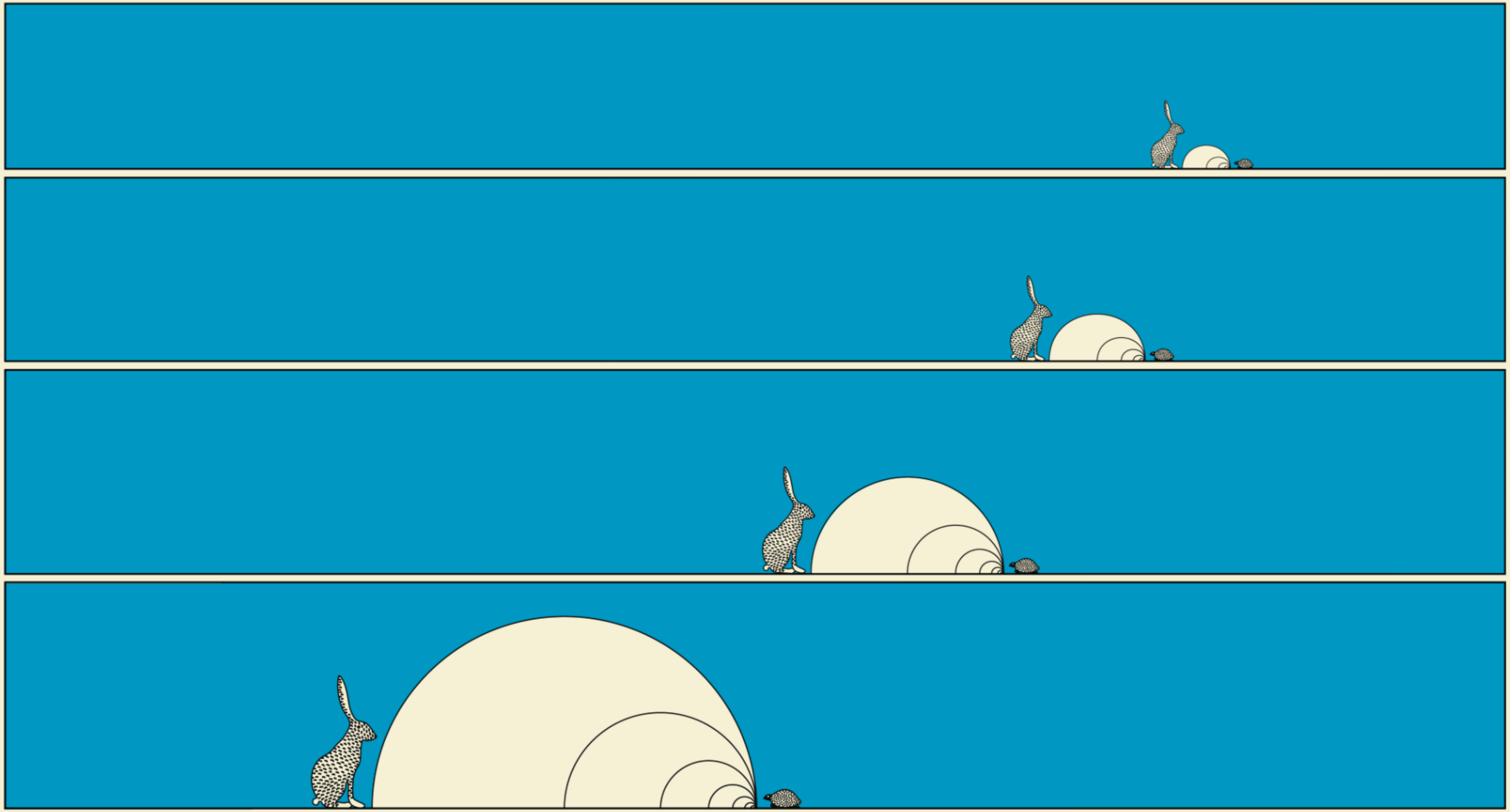

BLACK HOLES AND FUNDAMENTAL FIELDS, SCHOOL & WORKSHOP, LISBON, 1-5 JULY 2024

Symmetry and nonlinearity in black hole perturbation theory

Lam Hui

Columbia University

Work done with: Joyce, Penco, Santoni, Solomon; Beren, Sun; Lagos; Mittman, Lagos, Stein+; Podo

Ladder structure from SO(3,1) explains vanishing Love number

Quadratic QNMs are detectable and interesting

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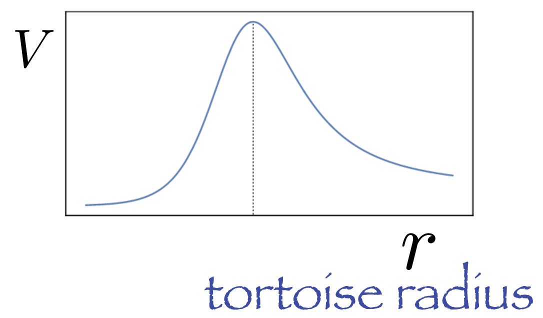

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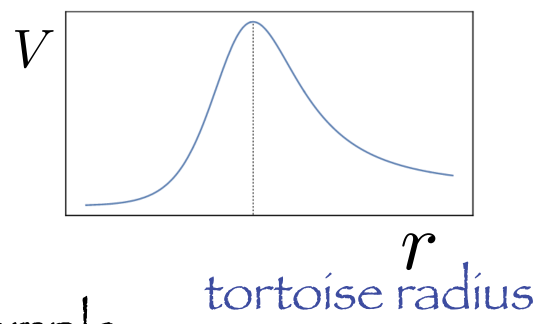

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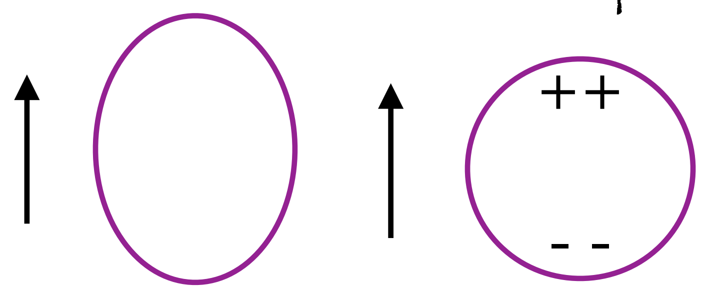

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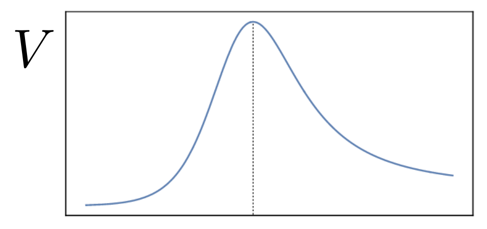

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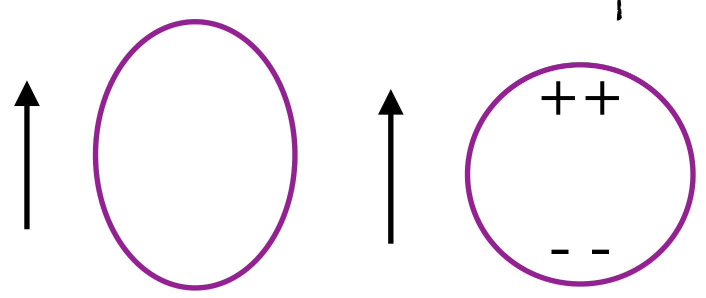

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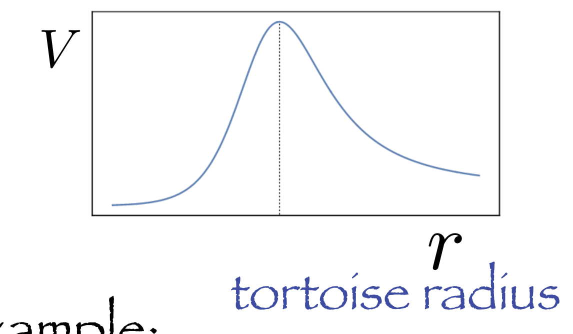

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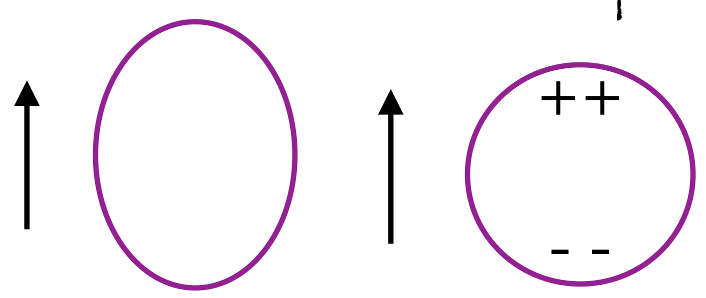

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Kerr metric: $$ds^2 = -\frac{\rho^2 - r_s r}{\rho^2} dt^2 - \frac{2 a r_s r \sin^2\theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta d\varphi^2$$

$$\underline{\Delta \equiv r^2 - r r_s + a^2} \qquad \rho^2 \equiv r^2 + a^2 \cos^2\theta$$

- Teukolsky equation:

$$\partial_r \left(\Delta \partial_r \phi_\ell^{(s)} \right) + s(2r - r_s) \partial_r \phi_\ell^{(s)} + \left(\frac{a^2 m^2 + i s (2r - r_s) a m}{\Delta} - (\ell - s)(\ell + s + 1) \right) \phi_\ell^{(s)} = 0$$

$\Delta \equiv r^2 - r r_s + a^2$ static: $\partial_t = 0$ expand in spin-weighted spherical harmonics

$a = \text{BH spin}$, $r_s = 2GM$, $s = 0, 1, 2$, $\ell, m = \text{ang. mom. quantum no.}$

$\phi_\ell^{(s)} = \text{Newman Penrose scalar} \sim \text{Weyl tensor projected onto null tetrad}$

electric and magnetic types contained in real and imaginary parts thereof

Spin ladder operators connect spin 0, 1, 2 perturbations i.e. by taking suitable derivatives, can raise and lower spin. This can be viewed as a generalization of the Teukolsky-Starobinsky identities. For static perturbations, something special happens.

The upshot: it is sufficient to understand the spin 0 case: $\underline{\Box \phi = 0}$

And for pedagogy , let's focus on Schwarzschild (will comment on Kerr).

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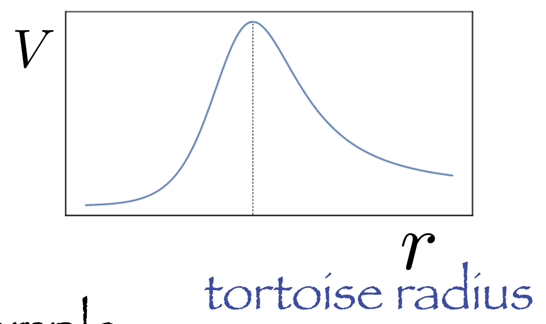

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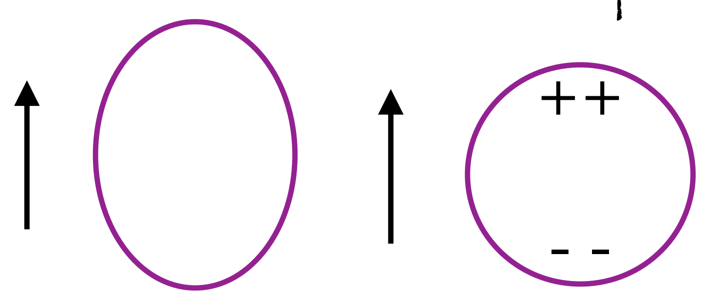

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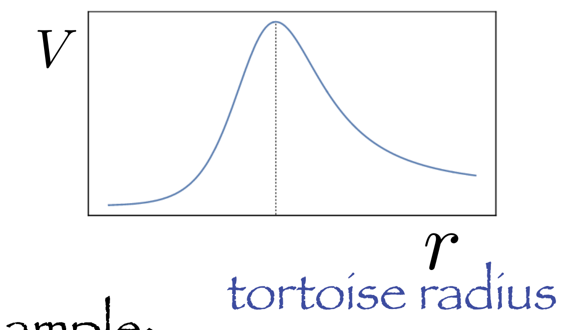

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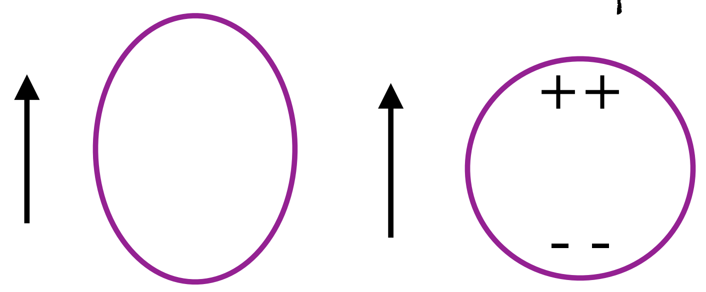

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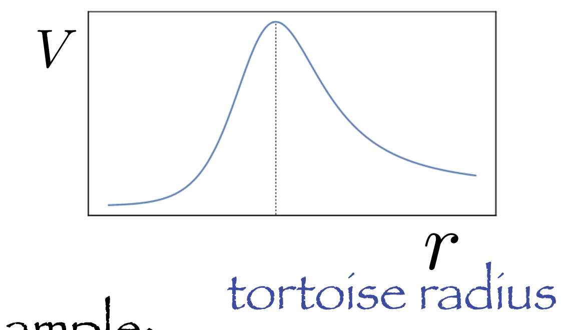

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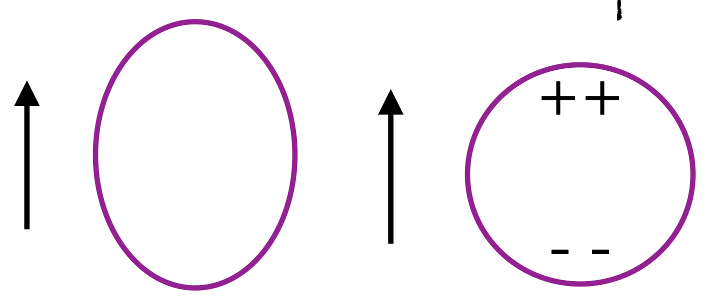

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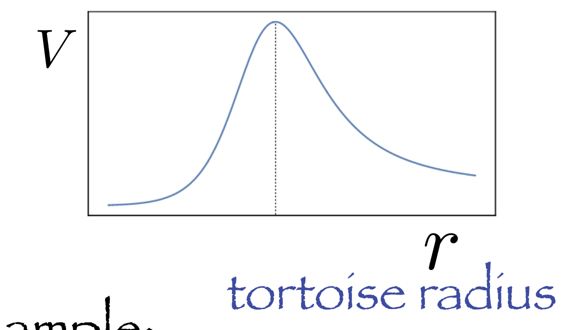

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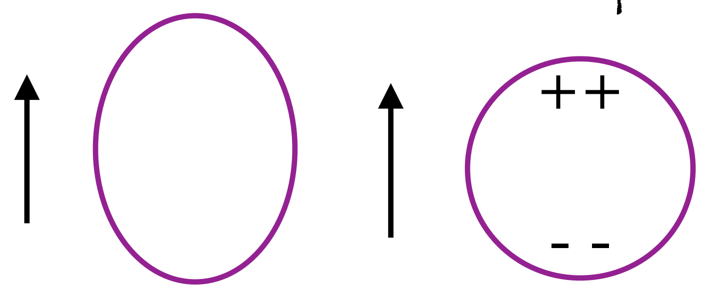

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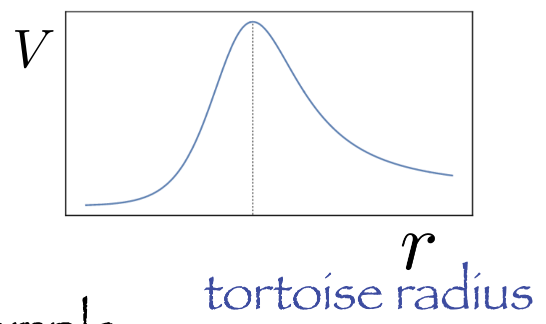

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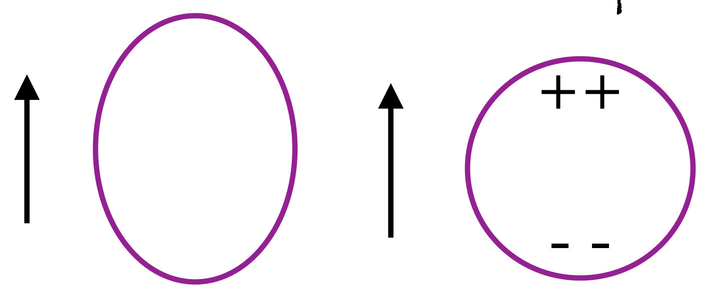

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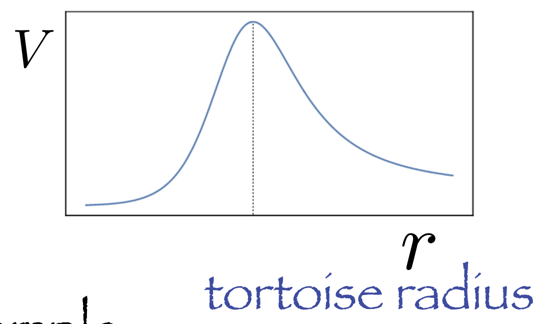

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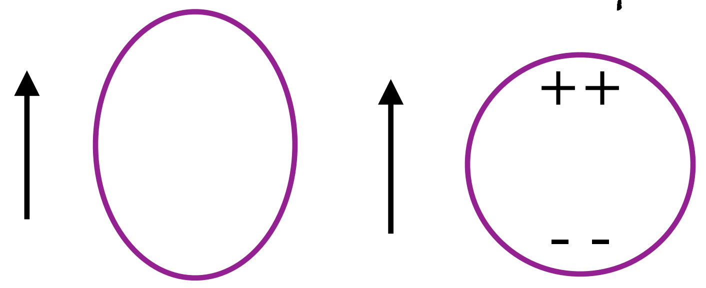

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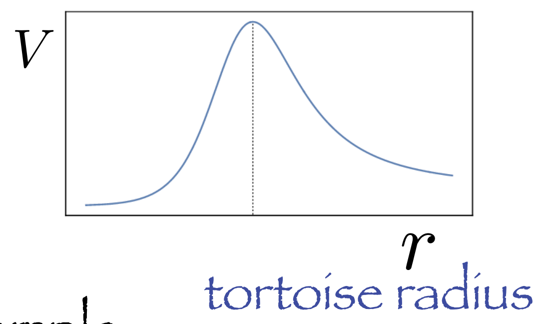

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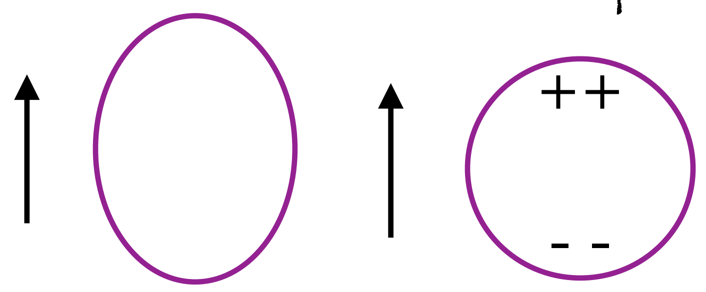

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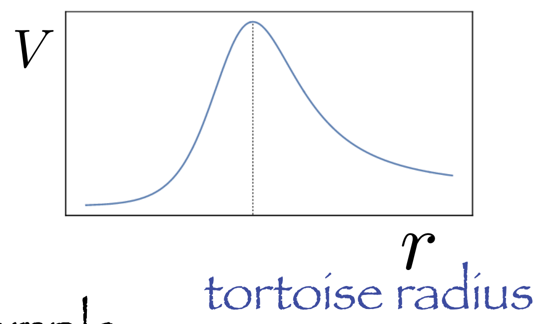

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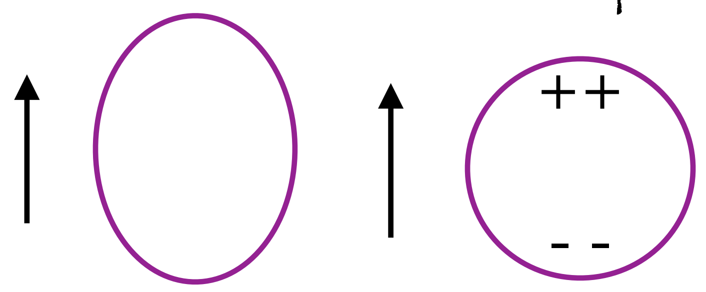

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- Extra comments: 1. SO(3,1) symmetry is exact (for static perturbations). 2. At large r, $\vec{K}$ generates special conformal trans.
3. Connection with reflection-less potential. 4. Higher symmetries: $\delta\phi_\ell = [D^+]^\ell \Delta\partial_r([D^-]^\ell \phi_\ell)$.

Symmetries: summary and further thoughts

- For static scalar on Schwarzchild: SO(3,1) exact symmetry. For Kerr: J_3, K_3 survive and are sufficient to generate the ladder structure for the Love number argument.

- For spin 1 and 2 perturbations: apply spin ladder.

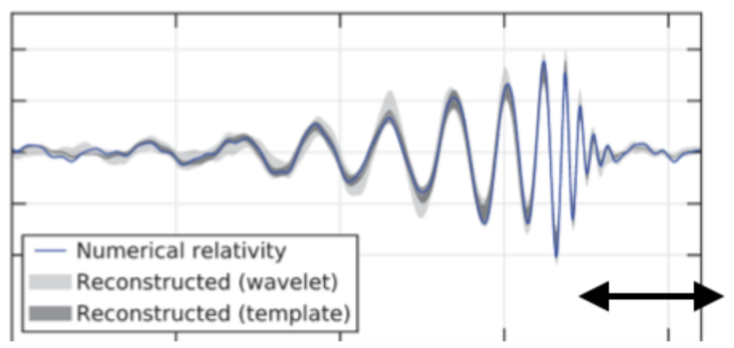
- For low frequency perturbations, SO(4,2) approximate symmetry is present, making possible the study of dynamical tidal response. An associated conserved charge is curiously connected with superradiance instability rate.

- For higher dimensional BH, ladder structure exists for $\hat{\ell} \equiv \dfrac{\ell}{D-3}$, and only for integer $\hat{\ell}$ can the "ground state" connection be made, which is crucial for the vanishing of the corresponding Love number.

- Nonlinear tidal response is being explored.

See also: Charalambous, Dubovsky, Ivanov; De Luca, Khoury, Wong; Riva, Santoni, Savic, Vernizzi; Rodriguez, Santoni, Solomon, Temoche; Rai, Santoni

Nonlinearity: quadratic quasi-normal modes



ring down

- Linear perturbation theory: $g = g_{\rm BH} + \phi \longrightarrow [\partial^2 + V]\phi \sim 0$

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  Write $\phi = \phi^{(1)} + \phi^{(2)} + \ldots \longrightarrow$ $[\partial^2 + V]\phi^{(1)} \sim 0$

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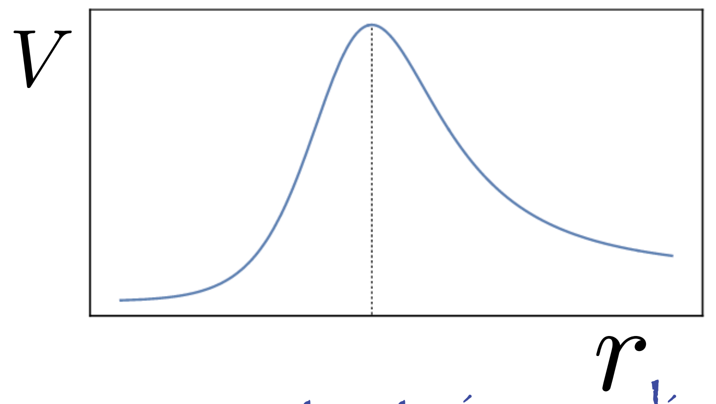
  Not surprising that: $\phi^{(1)}$ contains $e^{-i\omega_{220}t} \longrightarrow \phi^{(2)}$ contains $e^{-i\underline{2}\,\omega_{220}t}$

  linear QNM — quadratic QNM

  We searched for quadratic QNM in black hole merger simulations, and verified (220)x(220) amplitude goes as square of (220) amplitude (Mitman et al., Cheung et al. + much prior work).



tortoise radius

Expect quadratic quasi-normal modes from pairs of linear quasi-normal modes.

e.g. from pairs of ω_{220} we get quadratic $\omega = 2\omega_{220}$

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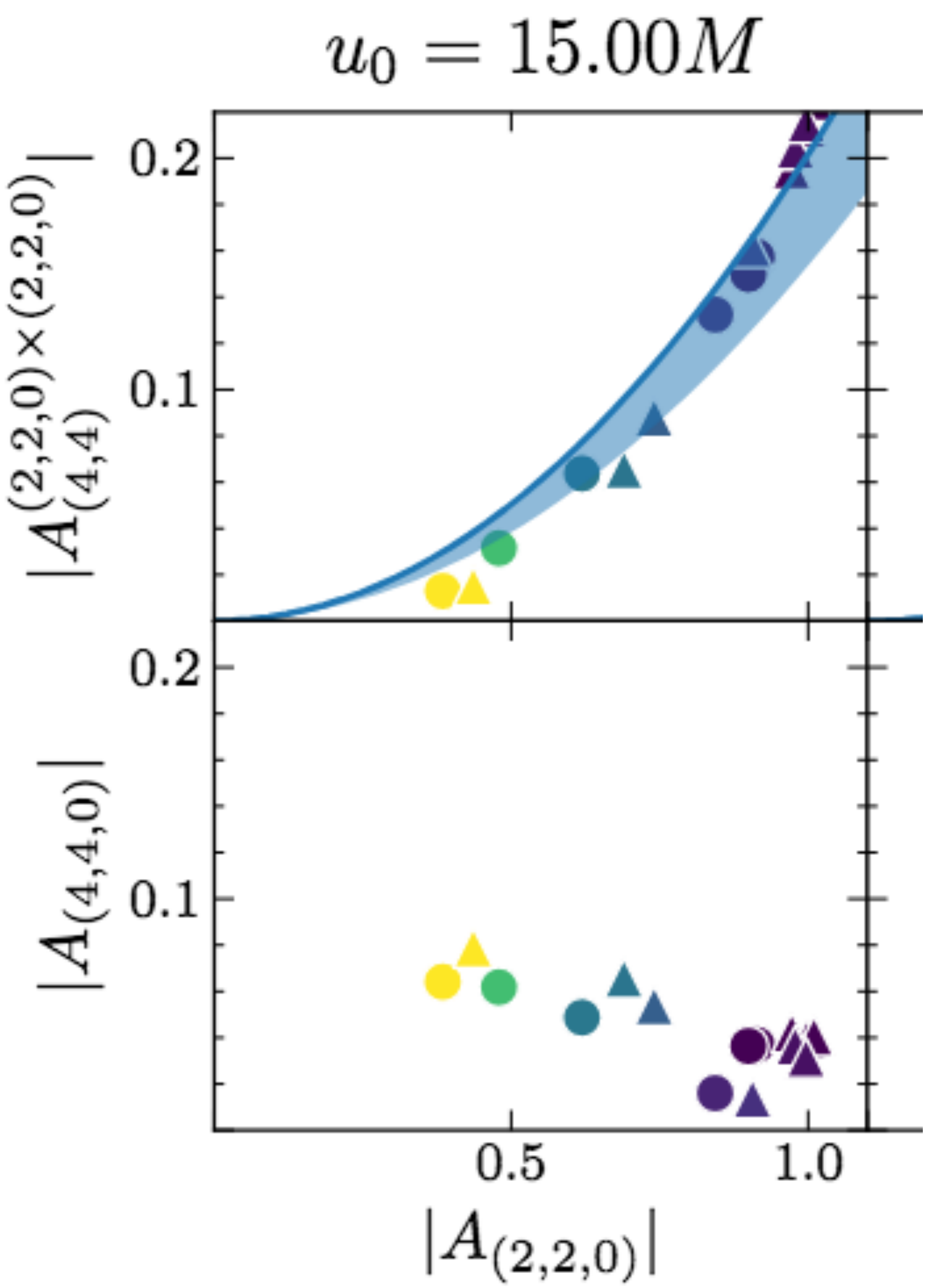
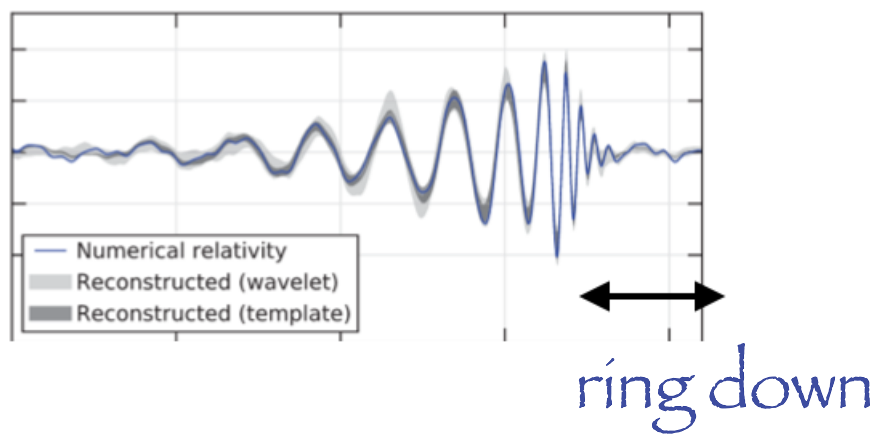


Colllaboration with Mitman, Lagos, Stein, Ma et al.

Colllaboration with Lagos.

See also Cheung et al.

Nonlinearity: quadratic quasi-normal modes



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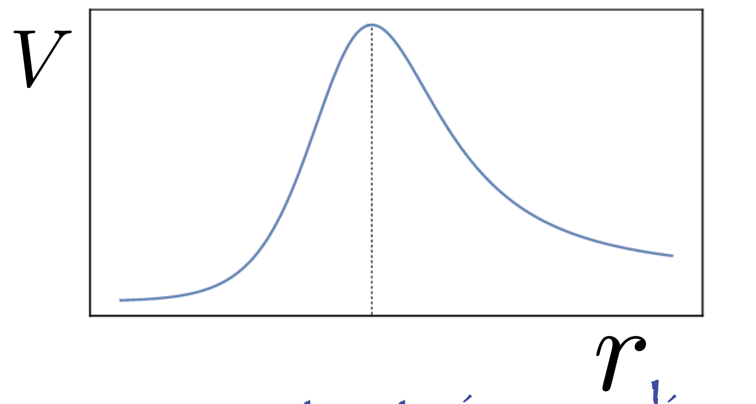

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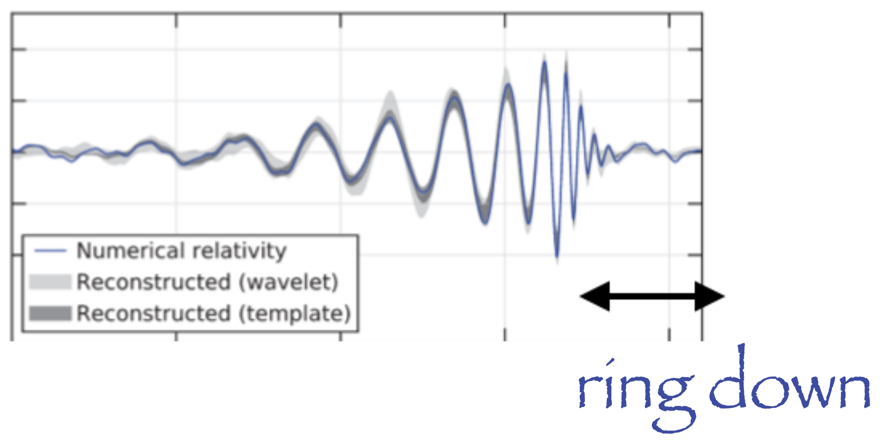

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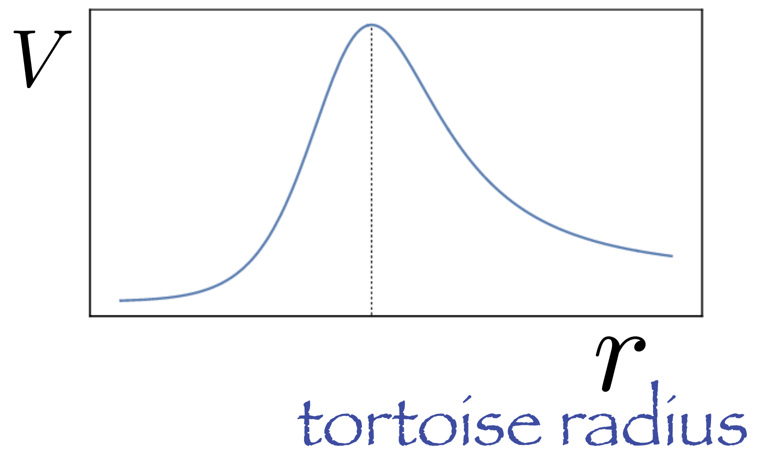

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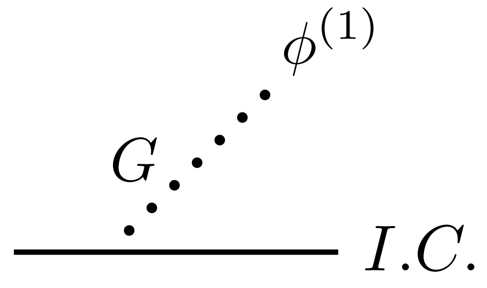
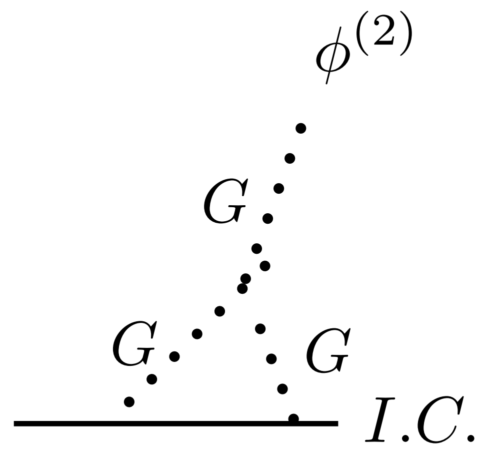
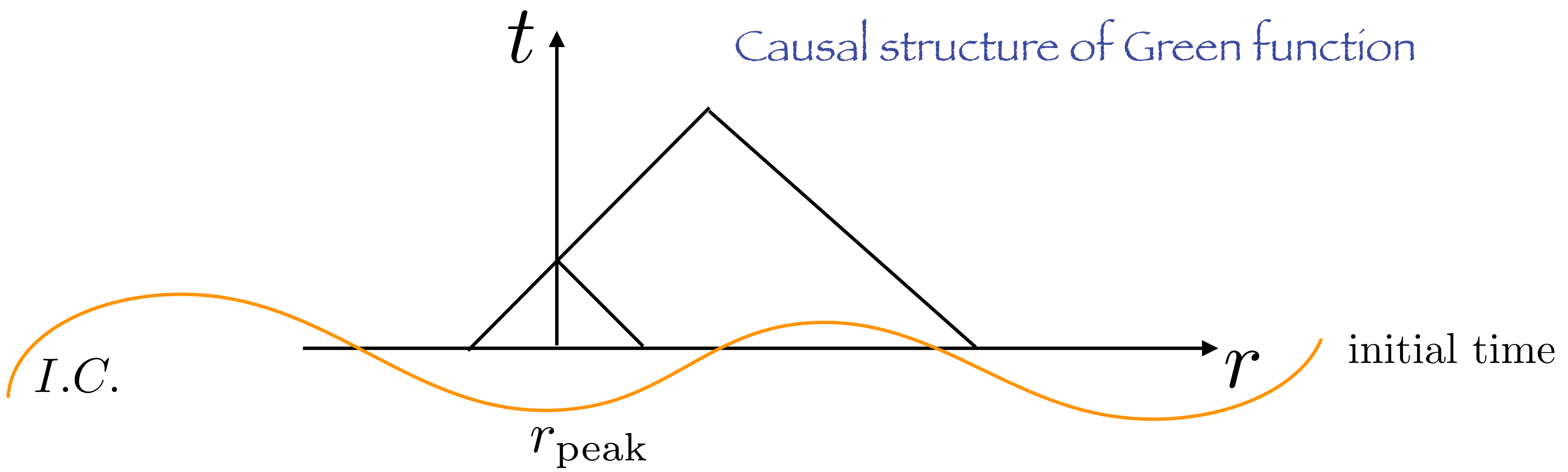
- Recall a QNM by itself is not a particularly physical quantity: $\phi^{(1)} \sim e^{-i\omega_{220}(t-r)}$ as $r \to \infty$ i.e. it diverges at large r.

  A more physical way to think about BH perturbations is via the Green function (Leaver, Andersson, Szpak...).

$$[\partial^2 + V]G = \delta_D$$

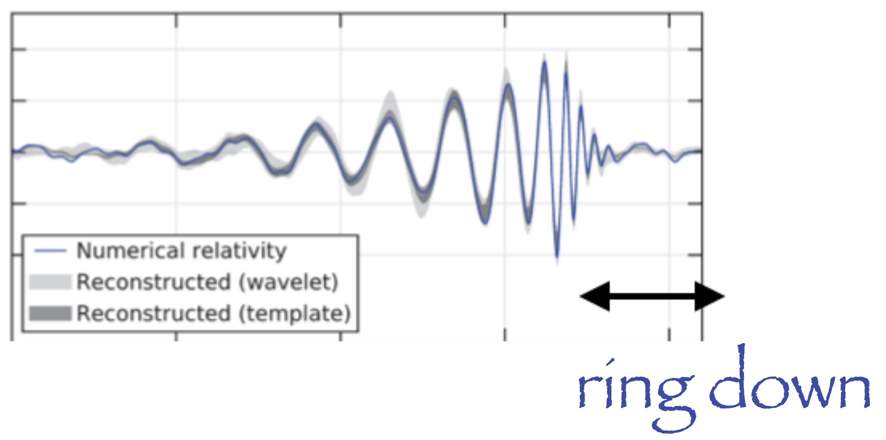






Many (known) implications: e.g. (1) ring-down waveform is not purely superposition of QNMs, (2) amplitude of QNMs is time dependent, (3) QNMs are generated from I.C. by scattering off potential barrier, etc.

Nonlinearity: quadratic quasi-normal modes



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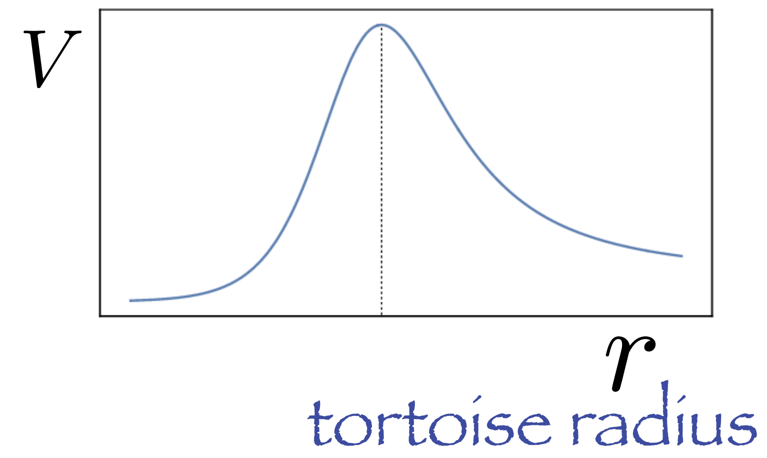

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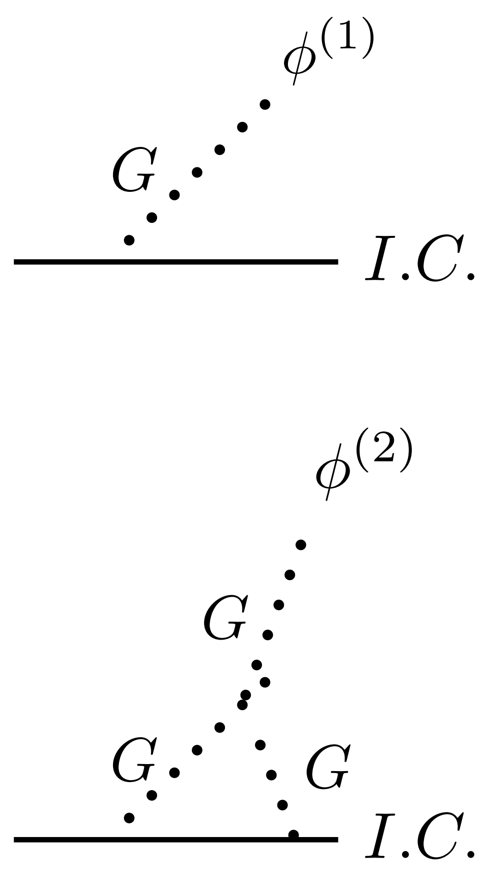

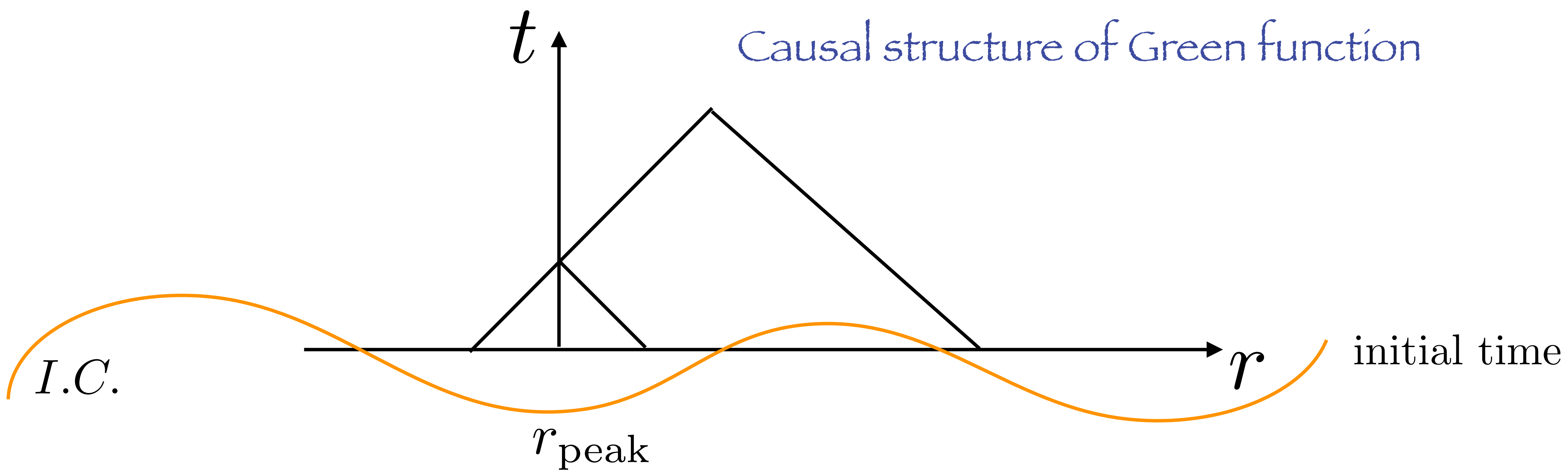

Many (known) implications: e.g. (1) ring-down waveform is not purely superposition of QNMs, (2) amplitude of QNMs is time dependent, (3) QNMs are generated from I.C. by scattering off potential barrier, etc.

- Why should we care? (1) more accurate modeling of the ring-down, (2) may get us closer to merger time, (3) relation between $\phi^{(2)}$ and $\phi^{(1)}$ (including amplitude) provides test of GR (e.g. Kehagias/Riotto, Redondo-Yuste+, Zhu+)

Ladder structure from SO(3,1) explains vanishing Love number

Quadratic QNMs are detectable and interesting

Symmetry and nonlinearity in black hole perturbation theory

Lam Hui

Columbia University

Work done with: Joyce, Penco, Santoni, Solomon; Beren, Sun; Lagos; Mittman, Lagos, Stein+; Podo