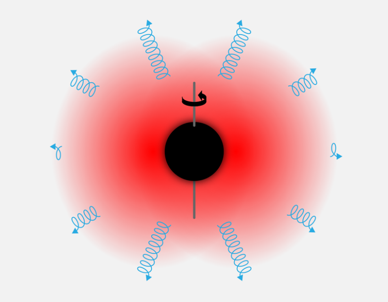
Lecture 1: **Superradiance**

Sam Dolan



University of Sheffield

New Horizons for Psi
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Superradiance

"Superradiance is a radiation enhancement process that involves dissipative systems."

Overview

1. A toy model for superradiance
2. Superradiance in a flowing fluid

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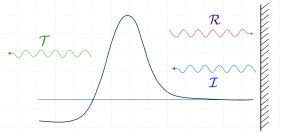
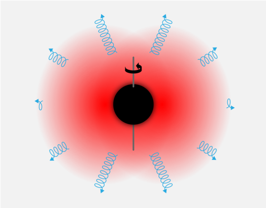
2. Superradiance in a flowing fluid
3. Superradiance from a charged black hole
4. Superradiance and the laws of black hole mechanics
5. Spinning black holes and the Penrose process
6. Scalar fields on Kerr spacetime.

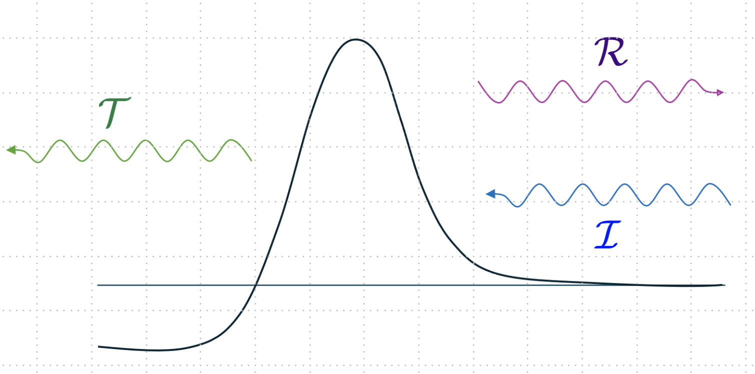




1. A toy model for superradiance

A Toy Model

A 1D scattering problem:



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$$\boxed{\left(\partial_\mu + iqA_\mu\right) \eta^{\mu\nu} \left(\partial_\nu + iqA_\nu\right) \Phi - V(x^\mu)\Phi = 0.}$$

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  - A monochromatic mode of the field: $\Phi(x^\mu) = \Phi(\mathbf{x})e^{-i\omega t}$.

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  - A 1D system:

$$\left\{\frac{d^2}{dx^2} + \left(\omega - q\varphi(x)\right)^2 - V(x)\right\}\Phi(x) = 0.$$

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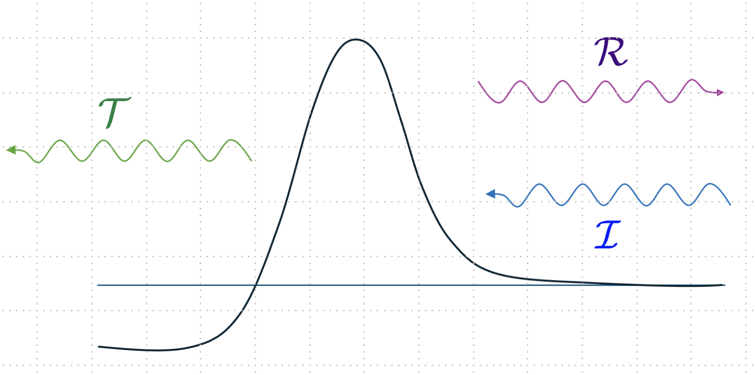
- **Asymptotic solutions:**

$$\begin{aligned} x \to +\infty : \qquad & \Phi(x) = \exp\left(\pm i\omega x\right), \\ x \to -\infty : \qquad & \Phi(x) = \exp\left(\pm i\widetilde{\omega} x\right). \end{aligned}$$

- Here $\widetilde{\omega} \equiv \omega - \omega_c$ with $\omega_c = q\varphi_0$.

A Toy Model

A 1D scattering problem:



$$\Phi_<(x) \equiv B_{\mathcal{T}}\, e^{-i\widetilde{\omega}x} \qquad \Phi_>(x) \equiv A_{\mathcal{I}}\, e^{-i\omega x} + A_{\mathcal{R}}\, e^{i\omega x}$$

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A Toy Model

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$$W[\Phi_1, \Phi_2] = \Phi_1 \frac{d\Phi_2}{dx} - \Phi_2 \frac{d\Phi_1}{dx}$$
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- If Φ is a solution then Φ^* is also a solution, because the potentials φ and $V(x)$ are **real**.
- Evaluating $W[\Phi, \Phi^*]$ in the limits $x \to \infty$ and $x \to -\infty$,
$$-i\widetilde{\omega} \left|B_{\mathcal{T}}\right|^2 = -i\omega \left(\left|A_{\mathcal{I}}\right|^2 - \left|A_{\mathcal{R}}\right|^2 \right)$$
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- Evaluating $W[\Phi, \Phi^*]$ in the limits $x \to \infty$ and $x \to -\infty$,
$$-i\widetilde{\omega} \left| B_{\mathcal{T}} \right|^2 = -i\omega \left(\left| A_{\mathcal{I}} \right|^2 - \left| A_{\mathcal{R}} \right|^2 \right)$$
- Hence
$$\mathcal{R} = 1 - \frac{\widetilde{\omega}}{\omega} \mathcal{T}, \qquad \mathcal{R} \equiv \left| \frac{A_{\mathcal{R}}}{A_{\mathcal{I}}} \right|^2, \quad \mathcal{T} \equiv \left| \frac{B_{\mathcal{T}}}{A_{\mathcal{I}}} \right|^2.$$

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- If $\widetilde{\omega}/\omega < 0$ then **superradiance**: $\mathcal{R} > 1$.

Exercise 1.1

Consider a **special case** of the toy model with a step-change in the electric potential and delta-function barrier:

$$\varphi(x) = \varphi_0 \, \Theta(-x), \qquad V(x) = V_0 \, \delta(x)$$

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$$A_{\mathcal{I}} + A_{\mathcal{R}} = B_{\mathcal{T}} \tag{1}$$
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2. Hence show that

$$\frac{A_{\mathcal{R}}}{A_{\mathcal{I}}} = \frac{\omega_c - iV_0}{\omega + \widetilde{\omega} + iV_0} \quad \text{and} \qquad 1 - \mathcal{R} = \frac{4\omega\widetilde{\omega}}{(\omega + \widetilde{\omega})^2 + V_0^2}. \tag{3}$$

2. Superradiance in a flowing fluid

Fluid flows

Visser '98 [paraphrased]

If a fluid is barotropic and inviscid, and the flow is irrotational then the equation governing the potential ψ for linearized perturbations in the flow (i.e. $\delta\mathbf{v} = -\nabla\psi$), is identical to a Klein-Gordon equation for a massless scalar field on a Lorentzian geometry. The effective metric for the geometry, $g_{\mu\nu}$, is determined by the fluid's properties and the background flow.

Fluid flows

- Consider an ideal fluid, which is vorticity-free, barotropic and inviscid, and flowing with a local velocity $\mathbf{v}_0(t, \mathbf{x})$
- Consider small perturbations (i.e. sound waves) in the flow, $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$ where $\delta\mathbf{v} = -\nabla\Phi$.

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on an **effective spacetime** with line element

$$ds^2 = -dt^2 + (d\mathbf{x} - \mathbf{v}_0 dt)^2$$

(Here I have set $c_s = 1$, $\rho = 1$).

Fluid flows: the draining bathtub

- A nice example is the **draining bathtub**: a 2D flow with velocity

$$\mathbf{v}_0 = -\frac{D}{r}\hat{\mathbf{r}} + \frac{C}{r}\hat{\boldsymbol{\phi}}$$

- C = Circulation rate, D = Draining rate.

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- Line element:

$$ds^2 = -dt^2 + \left(dr + \frac{Ddt}{r}\right)^2 + \left(rd\phi - \frac{Cdt}{r}\right)^2.$$

Exercise 1.2

$$ds^2 = -dt^2 + \left(dr + \frac{Ddt}{r}\right)^2 + \left(rd\phi - \frac{Cdt}{r}\right)^2$$

1. Show that the inverse metric in the coordinates $\{t, r, \phi\}$ is

$$g^{\mu\nu} = (\mathfrak{g})^{\mu\nu}, \quad \mathfrak{g} = \begin{pmatrix} -1 & \frac{D}{r} & -\frac{C}{r^2} \\ \cdot & 1 - \frac{D^2}{r^2} & \frac{CD}{r^3} \\ \cdot & \cdot & \frac{1-C^2/r^2}{r^2} \end{pmatrix}.$$

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2. The Jacobian matrix of a **coordinate transformation** $\tilde{x}^\mu(x^\nu)$ has components $(\mathfrak{J})^\mu{}_\nu \equiv \frac{\partial \tilde{x}^\mu}{\partial x^\nu}$. Use a matrix of the form

$$\mathfrak{J} = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}$$

to find a transformation $\tilde{\mathfrak{g}} \equiv \mathfrak{J}\, \mathfrak{g}\, \mathfrak{J}^T$ such that $\tilde{\mathfrak{g}}^{\tilde{t}r} = \tilde{\mathfrak{g}}^{r\tilde{\phi}} = 0$. Find α and β (which are functions of r).

Exercise 1.2 (ctd)

3. Show that in the new coordinate system

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} -1/f(r) & 0 & -\frac{C}{r^2 f(r)} \\ \cdot & f(r) & 0 \\ \cdot & \cdot & \frac{1-r_e^2/r^2}{r^2 f(r)} \end{pmatrix}, \qquad f(r) \equiv 1 - \frac{D^2}{r^2}.$$

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4. Starting with the Klein-Gordon equation in the new coordinate system $\{\tilde{t}, r, \tilde{\phi}\}$,

$$\Box \Phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) = 0,$$

and using the separation of variable $\Phi(x^\mu) = \frac{1}{\sqrt{r}} \Phi(r) e^{-i\omega\tilde{t} + im\tilde{\phi}}$, derive the radial equation

$$\left\{ \frac{d^2}{dx^2} + (\omega - \varphi(r))^2 - V(r) \right\} \Phi = 0,$$

where $dx/dr = 1/f$, $\varphi = Cm/r^2$, $V(r) = f(r) \left[\frac{(m^2 - 1/4)}{r^2} + \frac{5D^2}{4r^4} \right]$.

The Draining Bathtub

- $\varphi(r) = Cm/r^2 \Rightarrow \omega_c = m\Omega_H$ where $\Omega_H \equiv C/D^2$ is the **angular velocity** of the horizon at $r_h = D$.

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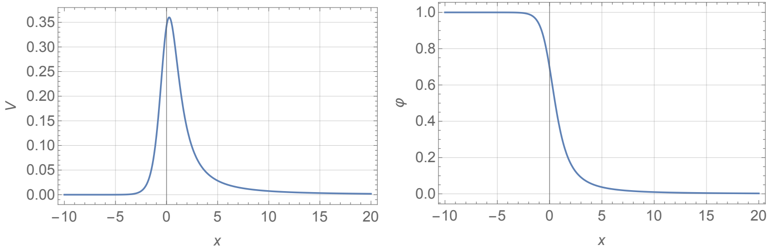
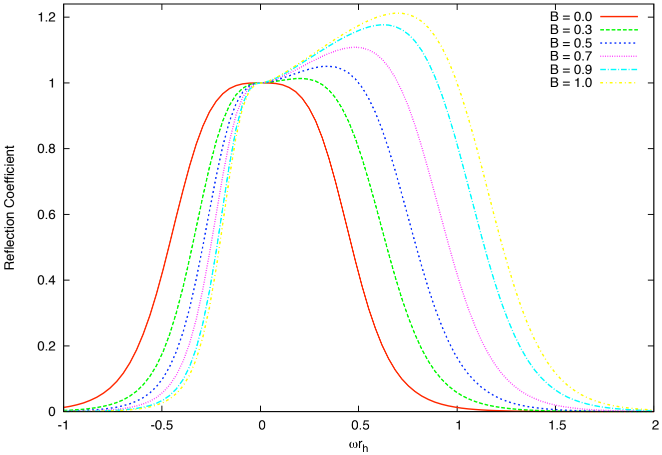
- Potentials as a function of x:

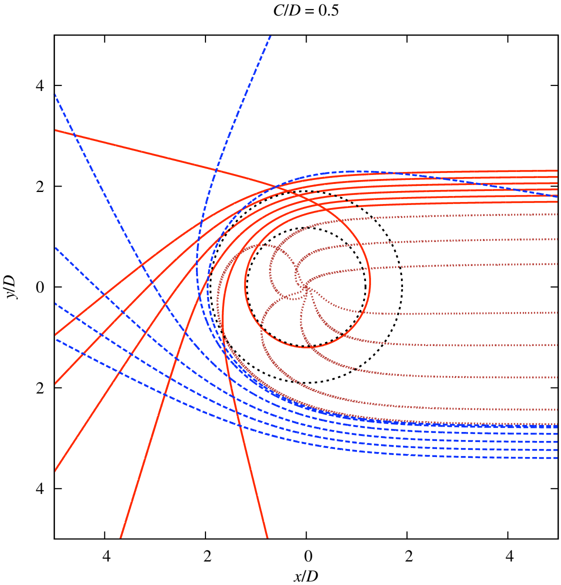


Superradiance for a draining bathtub

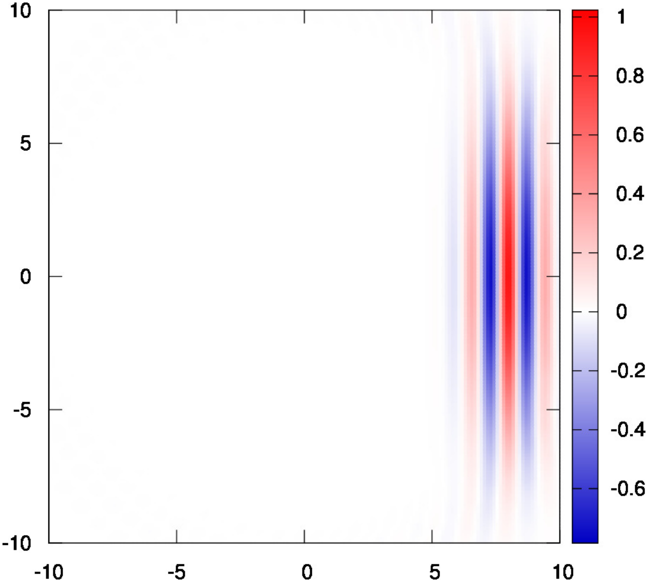


- $m = 1$ mode. Here $B = C$ (circulation rate) and $D = 1$.

Absorption by a draining bathtub: null geodesics



Wave scattering by a draining bathtub



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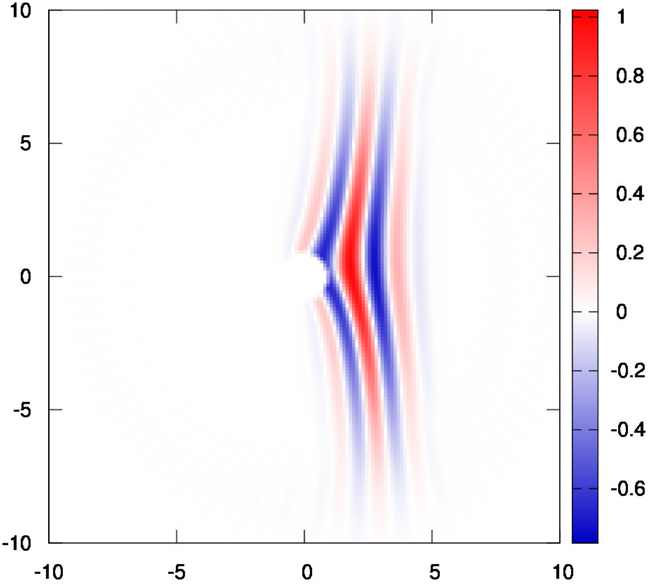

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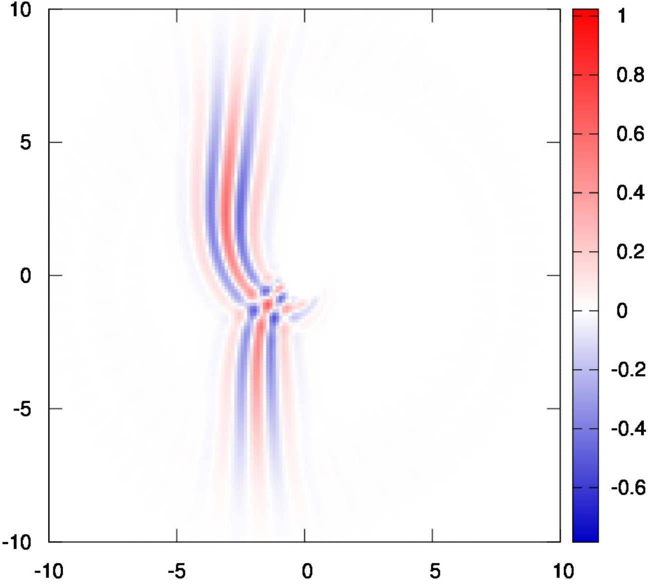

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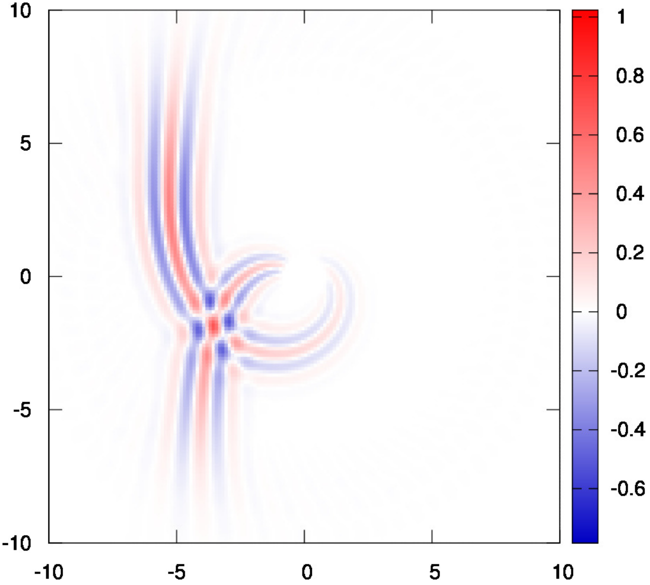

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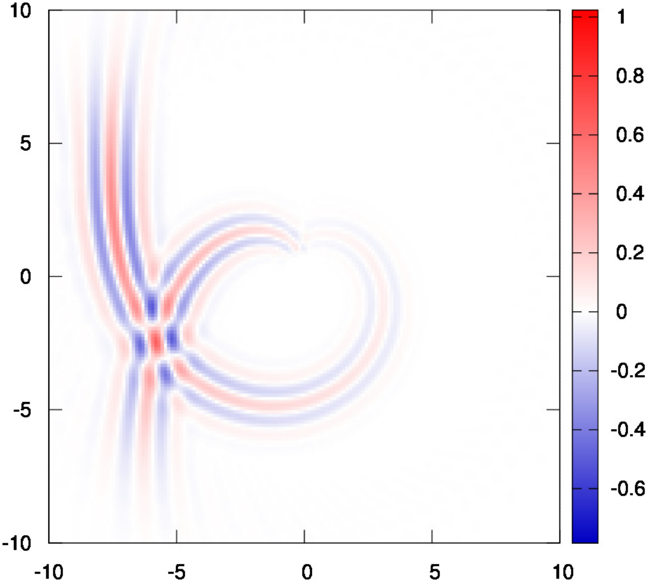
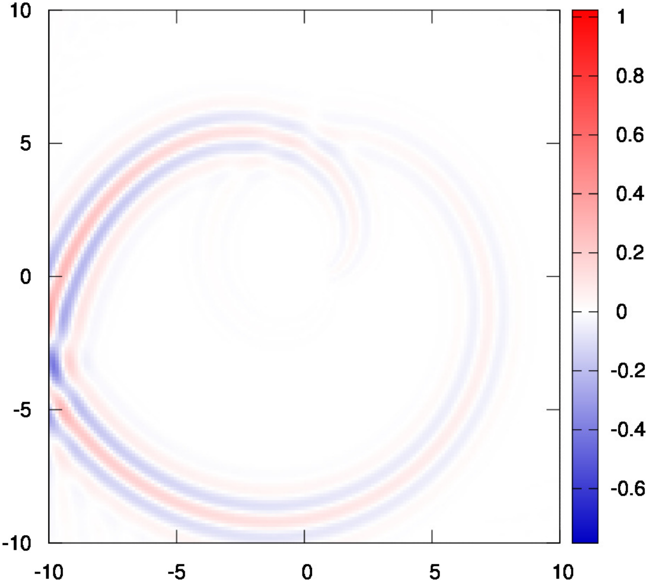


Wave scattering by a draining bathtub



Time-independent scattering: theory

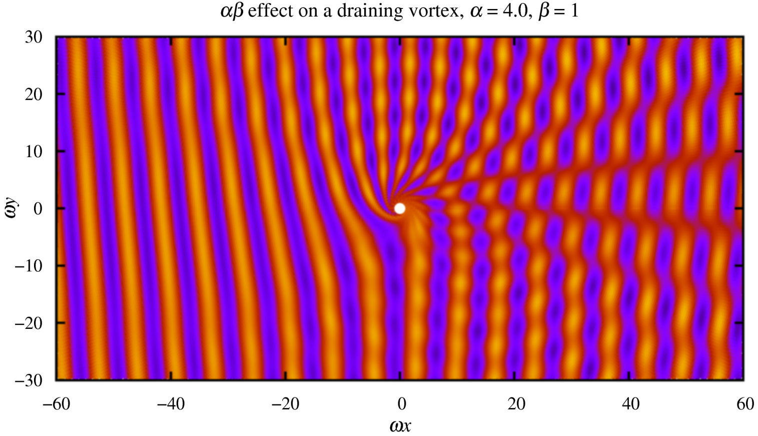


Figure: from SRD, Luis Crispino & Ednilton Oliveira (PLB, 2011).

A modified version of the **Aharonov-Bohm effect**.

Time-independent scattering: experiment

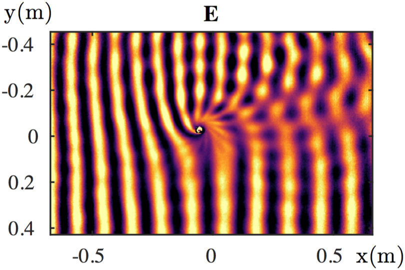


Figure: Nottingham wave tank experiment. Fig. 1 from Torres, Patrick, Coutant, Richartz, Tedford, and Weinfurtner, Nature Physics **13**, 833 (2017).

Superradiance in a wave tank

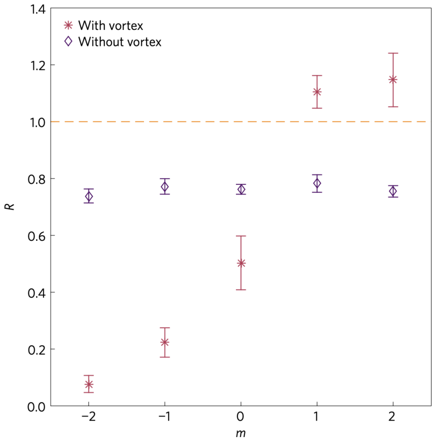


Figure: Reflection coefficients for different values of m, for the frequency $f = 3.7$Hz. Fig 3 from Torres *et al* (2017).

River model of black holes [Hamilton & Lisle 04]

FIG. 1: (Color online) The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall. Figure 1 of[5] presents a similar depiction.

- Imagine a 'spacetime flow' with local velocity $\mathbf{v}$

$$ds^2 = -c^2 dt^2 + (d\mathbf{x} - \mathbf{v} dt)^2$$

- Ergoregion where $|\mathbf{v}| \geq c$
- Apparent horizon where $|\hat{\mathbf{r}} \cdot \mathbf{v}| > c$

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- But, to describe **rotating** BH in this model also need local 'twist' bivector.
- cf. Lapse and shift

3. Superradiance from a charged black hole

Charged black holes and scalar fields

- The **Reissner-Nordström black hole** is a solution to the electrovacuum field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad \nabla_\nu F^{\mu\nu} = 0, \quad \nabla_{[\mu}F_{\nu\sigma]} = 0,$$

with $T_{\mu\nu} = F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{4}g_{\mu\nu}F_{\sigma\lambda}F^{\sigma\lambda}$.

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- The line element is

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

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  where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

- Horizons where $f(r) = 0$, at $r_\pm = M + \sqrt{M^2 - Q^2}$.

Charged black holes and scalar fields

- A (test) scalar field Φ with charge q satisfies the field equation

$$(\nabla_\mu + iqA_\mu)\,(\nabla^\mu + iqA^\mu)\,\Phi = 0.$$

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Exercise 1.3

Show that the above equation with the separation of variables $\Phi = \frac{1}{r}u(r)Y_{\ell m}(\theta, \phi)$ leads to the radial equation

$$\left\{ \frac{d^2}{dx^2} + (\omega - qQ/r)^2 - V(r) \right\} u = 0$$

where $dx/dr = 1/f(r)$ and

$$V(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r} \right).$$

Superradiance

$$\varphi(r) = qQ/r^2, \qquad V(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r}\right).$$

- Superradiance for $0 < \omega < \omega_c$ where $\omega_c = qQ/r_+$.
- The black hole loses mass **and** charge into the field.
- "But I thought nothing could come out of a black hole??"

4. Superradiance and the laws of black hole mechanics

The laws of black hole mechanics

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- Here A is the area of the black hole horizon, κ is surface gravity, Ω is its angular frequency, and Φ is its electrostatic potential.

Thermodynamics?

Bardeen, Carter & Hawking (1973)

"It can be seen that $\kappa/8\pi$ is analogous to temperature in the same way that A is analogous to entropy. It should however be emphasized that $\kappa/8\pi$ and A are distinct from the temperature and entropy of the black hole."

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$$\begin{aligned} c^2 dM &= \left(\frac{\hbar}{2\pi k_B c}\kappa\right) d\left(\frac{k_B c^3}{G\hbar}\frac{A}{4}\right) + \Omega dJ + \Phi dQ \\ dU &= \qquad T \qquad\quad dS \qquad\quad - pdV + \mu dN. \end{aligned}$$

- Horizon area A $\Leftrightarrow$ Entropy S?

Entropy

- Bekenstein-Hawking entropy:

$$S = \frac{k_B c^3}{G\hbar} \frac{A}{4} \qquad \approx 10^{54} \left(\frac{M}{M_\odot}\right)^2 \mathrm{JK}^{-1}.$$

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- A Universe whose entropy is **dominated** by black holes!

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The Sun	$\sim 10^{35}$
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- Should we believe this?
  What are the microstates of the black hole?

Superradiance from black hole mechanics

- First law:

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ.$$

Rearranging,

$$\Rightarrow \quad dA = \frac{8\pi}{\kappa} \left(1 - \Omega \frac{dJ}{dM} - \Phi \frac{dQ}{dM} \right) dM.$$

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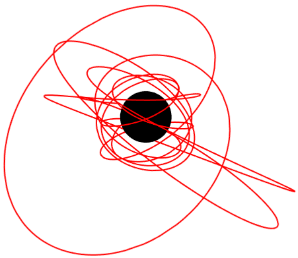
$$0 < \omega < \omega_c, \qquad \boxed{\omega_c \equiv m\Omega + q\Phi.}$$

5. Spinning black holes and the Penrose process

Kerr spacetime



A timelike geodesic around a Kerr black hole.

Image: Black Hole Perturbation Toolkit (`bhptoolkit.org`).

- Astrophysical black holes are unlikely to be significantly charged, but those formed in binaries are likely to be rotating.
- Characterised by the **spin parameter** $a \equiv J/M$.

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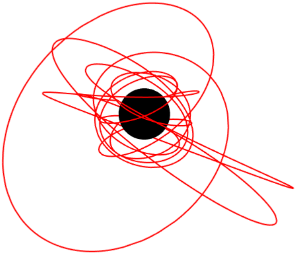

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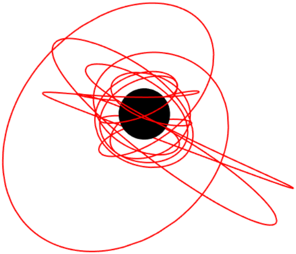

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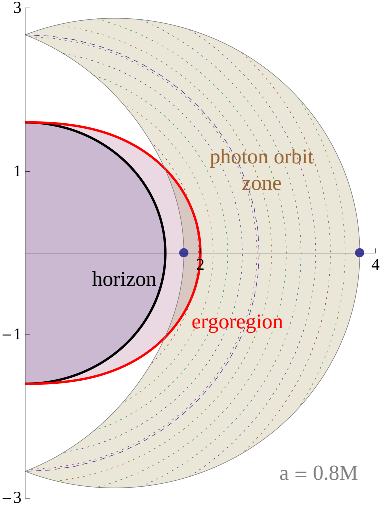
- Characterised by the **spin parameter** $a \equiv J/M$.
- $a = 0 \quad \Rightarrow$ Schwarzschild BH.
- $a = M \Rightarrow$ Extremal BH ($\kappa \to 0$).
- Cosmic censorship: singularities are hidden by event horizons $\Rightarrow -M < a < M$.

Symmetries of Kerr spacetime



- Stationary and axisymmetric ⇒ Two Killing vectors: $\nabla_{(\mu} X_{\nu)} = 0$. Conserved energy $\mathcal{E} = -X^{\mu}_{(t)} u_{\mu}$ and az. angular momentum $\mathcal{L} = X^{\mu}_{(\phi)} u_{\mu}$.

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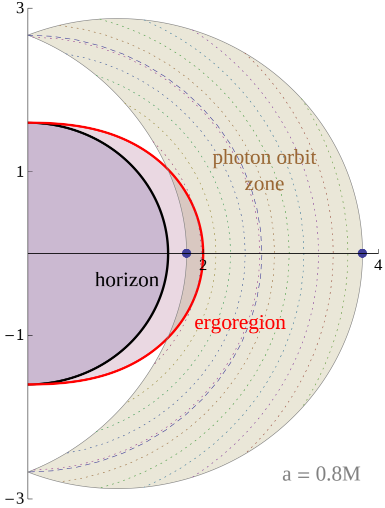

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- Killing **tensor** $\nabla_{(\mu} K_{\nu\sigma)} = 0$ ⇒ Carter constant $\mathcal{K} \equiv K_{\mu\nu} u^\mu u^\nu$ ⇒ Geodesic motion is **integrable**.

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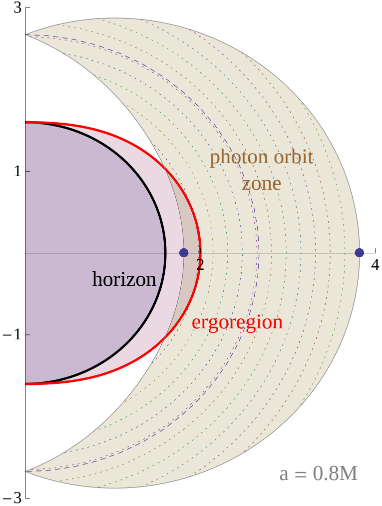

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  $\Rightarrow$ Geodesic motion is **integrable**.
- **Commuting operators:**
  $[X^{\mu}\nabla_{\mu}, \Box] = 0$
  $[\nabla_{\mu}K^{\mu\nu}\nabla_{\nu}, \Box] = 0$.
  Separability of wave equations.

Stationary limit surface and ergoregion

- The **stationary limit surface**is a surface on which the time-translation Killing vector $X^{\mu}_{(t)} = [1, 0, 0, 0]$ is **null**

$$X^{\mu} X_{\mu} = 0.$$

- Inside the SL surface is the **ergoregion** in which $X^{\mu}_{(t)}$ is **spacelike**.

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- Inside the ergoregion, all timelike worldlines have $d\phi / d\tau > 0$, i.e., motion is in the same direction as the spin of the BH.
- For a spinning black hole, the ergoregion extends outside the horizon at $r_{+} = M + \sqrt{M^2 - J^2}$ to the SL surface at $r = 2M$.

Exercise 1.4

Two vectors A^μ and B^μ are defined at a point in a Lorentzian spacetime. Show that:

- If A^μ and B^μ are both timelike & future-pointing, then $A^\mu B_\mu < 0$.
- If A^μ is timelike f-p and B^μ is spacelike then $A^\mu B_\mu$ can take either sign.

The Penrose process

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- Hence, inside the ergoregion – where $X^\mu_{(t)}$ is spacelike – particles can have **negative energies**.
- A Penrose process takes advantage of this fact to extract energy from the black hole.

The Penrose process

- Suppose a particle, coming from infinity, enters the ergoregion of a black hole.

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- By conservation of energy,

$$\mathcal{E}_1 = \mathcal{E}_A + \mathcal{E}_B \quad \Rightarrow \quad \mathcal{E}_B = \mathcal{E}_1 - \mathcal{E}_A.$$

- Since $\mathcal{E}_A < 0$, the escaping particle (B) has **more** energy than the incident particle (1).

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- Since $\mathcal{E}_A < 0$, the escaping particle (B) has **more** energy than the incident particle (1).
- In a Penrose process, **energy is extracted from the black hole**.

The Penrose process

- Suppose a particle, coming from infinity, enters the ergoregion of a black hole.
- Particle 1 now fissions into two particles, A and B.
  Suppose particle A has **negative energy**.
- Particle A cannot leave the ergosphere. But particle B can escape to infinity. (It is necessary to show that this can be achieved).
- By conservation of energy,

$$\mathcal{E}_1 = \mathcal{E}_A + \mathcal{E}_B \quad \Rightarrow \quad \mathcal{E}_B = \mathcal{E}_1 - \mathcal{E}_A.$$

- Since $\mathcal{E}_A < 0$, the escaping particle (B) has **more** energy than the incident particle (1).
- In a Penrose process, **energy is extracted from the black hole**.
- The Penrose process is the particle version of superradiance for waves.

6. Scalar fields on Kerr spacetime

Scalar field on Kerr spacetime

- Let's consider a scalar field with a mass μ, governed by the Klein-Gordon equation

$$\Box\Phi - \mu^2\Phi = 0.$$

- This equation is **separable** on Kerr spacetime (Brill *et al* 1972), despite the lack of spherical symmetry.

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- This equation is **separable** on Kerr spacetime (Brill *et al* 1972), despite the lack of spherical symmetry.
- We will use Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$, and the inverse metric in the form

$$g^{\mu\nu} = \frac{1}{\Sigma}\left(\Delta l_+^{(\mu} l_-^{\nu)} + m_+^{(\mu} m_-^{\nu)}\right).$$

- Here $\{l_+^\mu, l_-^\mu, m_+^\mu, m_-^\mu\}$ is a complex null tetrad, such that $l_\pm^\mu$ align with the **principal null directions** of the spacetime.

Scalar field on Kerr spacetime: separability

$$g^{\mu\nu} = \frac{1}{\Sigma}\left(\Delta l_+^{(\mu} l_-^{\nu)} + m_+^{(\mu} m_-^{\nu)}\right)$$

- $l_\pm^\mu$ and Δ are functions of r **only**, and $l_\pm^r = 1$, $l_\pm^\theta = 0$.

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- Here $\Sigma = r^2 + a^2 \cos^2\theta$ and the metric determinant is given by $\sqrt{-g} = \sin\theta\Sigma$.

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Exercise 1.5

Using the above, show that $\Box\Phi - \mu^2\Phi = 0$ is equivalent to

$$\left\{\mathcal{D}\Delta\mathcal{D}^\dagger + \mathcal{D}^\dagger\Delta\mathcal{D} - 2\mu^2 r^2\right\}\Phi + \left\{\mathcal{L}_1\mathcal{L}_0^\dagger + \mathcal{L}_1^\dagger\mathcal{L}_0 - 2\mu^2 a^2\cos^2\theta\right\}\Phi = 0$$

with $\mathcal{D} = l_+^\mu\partial_\mu$, $\mathcal{D}^\dagger = l_-^\mu\partial_\mu$, $\mathcal{L}_n^\dagger = m_+^\mu\partial_\mu + n\cot\theta$, $\mathcal{L}_n = m_-^\mu\partial_\mu + n\cot\theta$.

Scalar field on Kerr spacetime: separability

With the previous result we have a separation of variables

$$\Phi = R(r)S(\theta)e^{-i\omega t + im\phi}$$

with some angular eigenvalue λ such that

$$\frac{1}{2}\left\{\mathcal{L}_1\mathcal{L}_0^\dagger + \mathcal{L}_1^\dagger\mathcal{L}_0 - 2\mu^2 a^2\cos^2\theta\right\}S(\theta) = -\lambda S(\theta).$$

The null tetrad is

$$\begin{aligned} l_\pm^\mu &= \left[\pm\frac{r^2+a^2}{\Delta}, 1, 0, \pm\frac{a}{\Delta}\right] \\ m_\pm^\mu &= \left[\pm ia\sin\theta, 0, 1, \pm\frac{i}{\sin\theta}\right], \end{aligned}$$

where $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$.

Scalar field on Kerr spacetime: separability

Exercise 1.6

1. Show that, when acting on a harmonic function $\propto e^{-i\omega t + im\phi}$, the derivative operators are $\mathcal{D} = \partial_r - iK/\Delta$ and $\mathcal{D}^\dagger = \partial_r + iK/\Delta$, where $K \equiv \omega \mathfrak{R}^2 - am$ and $\mathfrak{R}^2 \equiv r^2 + a^2$.
2. Hence show that the radial equation is

$$\left\{ \Delta \partial_r \Delta \partial_r + K^2 - \Delta(\lambda + \mu^2 r^2) \right\} R(r) = 0.$$

3. By defining $R(r) = u(r)/\mathfrak{R}$, show that the radial equation takes the canonical form

$$\left\{ \frac{d^2}{dx^2} + \left(\omega - \frac{am}{r^2 + a^2} \right)^2 - V(r) \right\} u(r) = 0,$$

where $dx/dr = (r^2 + a^2)/\Delta$ and $V(r)$ is a function that you should determine.