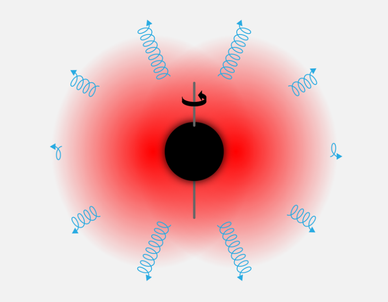
Lecture 2: **Superradiant Instabilities**

Sam Dolan

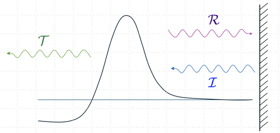


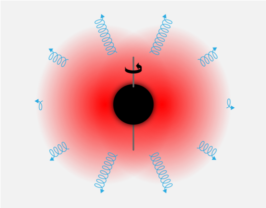
University of Sheffield

New Horizons for Psi
1st July, 2024.

Overview

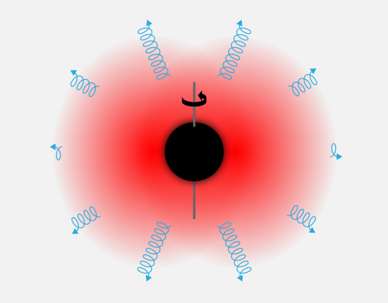
1. Key paper: "Floating Orbits, Superradiant Scattering and the Black-hole Bomb"
2. Toy model with a mirror.
3. Quasi-bound states of the Kerr black hole.
4. Massive (dark) photons and gravitons?
5. Black holes as particle detectors: proposals.





Instabilities

Superradiance + Confinement $\Rightarrow$ Instability



1. “Floating Orbits, Superradiant Scattering and the Black-hole Bomb”

W. H. Press and S. A. Teukolsky, Nature **238**, 211 (1972).

Key paper (1972)

Floating Orbits, Superradiant Scattering and the Black-hole Bomb

Penrose[1] and Christodoulou[2] have shown how, in principle, rotational energy can be extracted from a black hole by orbiting and fissioning particles. Recently, Misner[3] has pointed out that waves can also extract rotational energy ("superradiant scattering" in which an impinging wave is amplified as it scatters off a rotating hole). As one application of superradiant scattering, Misner has suggested the possible existence of "floating orbits", that is, orbits in which a particle radiatively extracts energy from the hole at the same rate as it radiates energy to infinity; thereby it experiences zero net radiation reaction.

Here we point out a second application of superradiant scattering which we call the "black-hole bomb". We also

Key paper

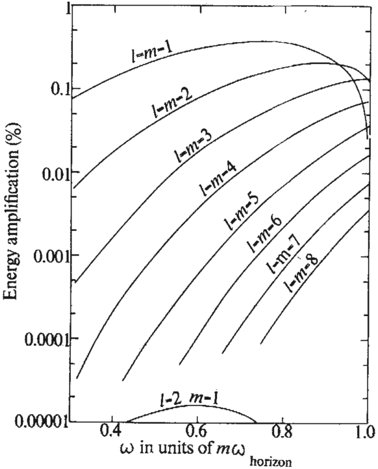
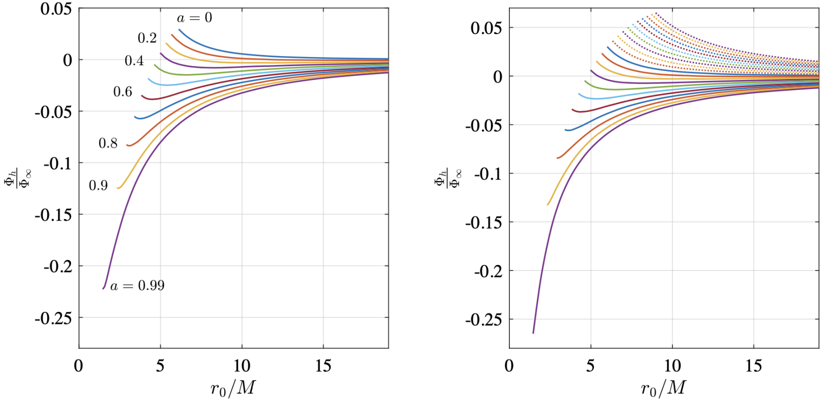


Fig. 1 Superradiant scattering of scalar radiation by maximally-rotating black hole. Radiation modes with axial eigenvalue $m > 0$ and angular frequency $\omega < m\omega_{\text{horizon}}$ are amplified by the hole, not absorbed by it. The fractional wave energy added by the hole is here shown as a function of wave frequency for the most favourable modes.

Floating orbits?

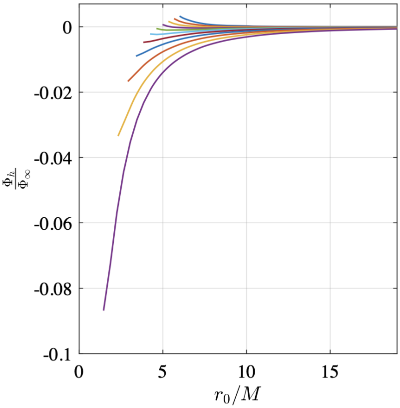


(a) Scalar field $s = 0$.

(b) Electromagnetic field $s = 1$.

- Floating orbit $\Leftrightarrow$ ratio of fluxes $\Phi_h/\Phi_\infty = -1$
- The $m = \ell = \min(1, |s|)$ mode dominates the radiation.

Floating orbits?



(c) Gravitational field $s = 2$.

- Insufficient superradiant flux to balance the radiation at infinity.
- Floating orbits do **not** arise.

A black hole bomb?

Press and Teukolsky 1972

"To illustrate, in a rather speculative vein, we propose the **black hole bomb**: locate a rotating black hole and construct a spherical mirror around it. The mirror must reflect low-frequency radio waves with reflectivity $\gtrsim 99.8\%$, so that in one reflexion and subsequent superradiant scattering there is a net amplification. The system is then unstable against a numer of exponentially growing electromagnetic modes which will be initiated by random "seed fields" (thermal noise).

"Others may care to speculate on the possibility that **nature provides her own mirror**. The amplified wave frequencies are far below the plasma frequency of the interstellar medium, so that waves would reflect off the boundary of an evacuated cavity surrounding the hole."

2. A toy model with a mirror

A Toy Model

Recall from Lecture 1 the toy model problem with the step in the electric potential, and a delta-function barrier:

$$\left\{ \frac{d^2}{dx^2} + (\omega - q\varphi(x))^2 - V(x) \right\} \Phi(x) = 0.$$

$$\varphi(x) = \varphi_0 \, \Theta(-x), \qquad V(x) = V_0 \, \delta(x).$$

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We derived that

$$\alpha \equiv \frac{A_{\mathcal{R}}}{A_{\mathcal{I}}} = \frac{\omega_c - iV_0}{\omega + \widetilde{\omega} + iV_0}$$

where $\widetilde{\omega} = \omega - \omega_c$ with $\omega_c \equiv q\varphi_0$.

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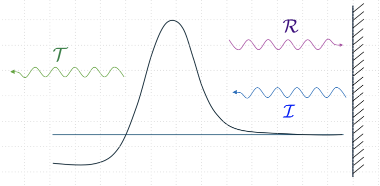
Now let's put a **mirror** at $x = x_0 > 0$.



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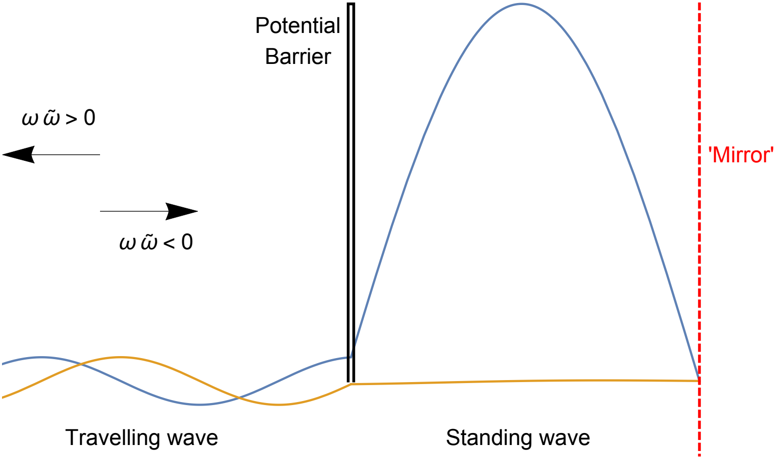

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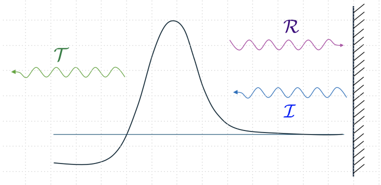

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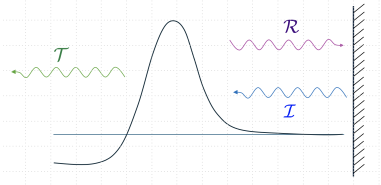

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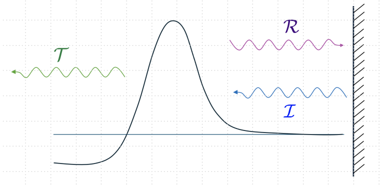

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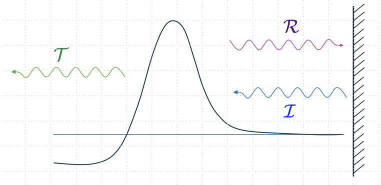

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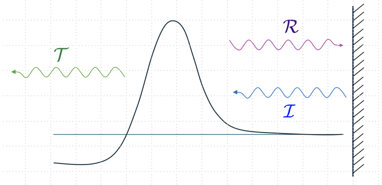

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$$e^{-2i\omega_r x_0} \approx 1 \quad \Rightarrow \quad \boxed{\omega_r \approx \frac{\pi n}{x_0}, \quad n \in \mathbb{Z}}$$

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- $\omega_r(\omega_r - \omega_c) < 0 \Rightarrow$ superradiance $\Rightarrow \omega_i > 0 \Rightarrow$ exponential growth.

3. Quasi-bound states of the Kerr black hole

Scalar field on Kerr spacetime

- Recall from exercise 1.6 that $\Box\Phi - \mu^2\Phi = 0$ yields the radial equation

$$\left\{ \frac{d^2}{dx^2} + \left(\omega - \frac{ma}{r^2 + a^2}\right)^2 - V(r) \right\} u(r) = 0,$$

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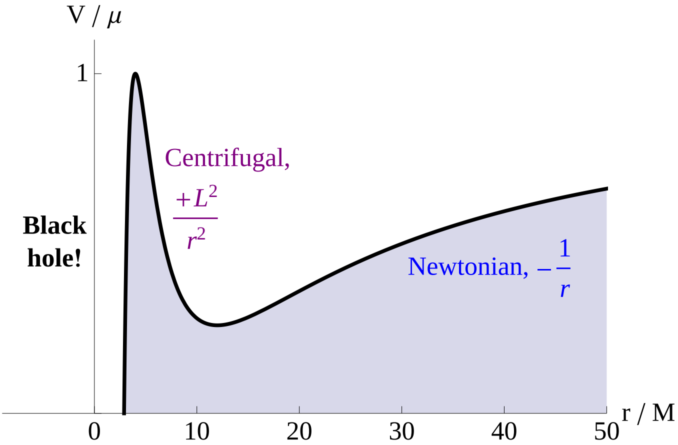
- Low-frequency waves $0 < \omega < m\Omega_H$ are superradiant.

Scalar field on Kerr spacetime

- We do not need a mirror for confinement; with a field mass μ some modes of the field are bound to the black hole.



Bound states of scalar field

- A scalar field Φ satisfying $\Box\Phi - \mu^2\Phi = 0$ which is regular on $\mathcal{H}^+$ and as $r \to \infty$ has a **discrete spectrum** of complex frequencies

$$\omega = \hat{\omega} + i\nu$$

labelled by azimuthal m and total l ang. mom., and overtone $\hat{n}$.

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- In limit $\alpha \equiv M\mu \ll l$, there is a **hydrogenic spectrum** with fine structure corrections:

$$\frac{\hat{\omega}}{\mu c^2} \approx 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{8n^4} + \frac{(2l - 3n + 1)\alpha^4}{n^4(l + 1/2)} + \frac{2am/M\alpha^5}{n^3 l(l + 1/2)(l + 1)} + \ldots$$

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- For Schwarzschild BH, all states decay $\nu < 0$.
- For Kerr BH, states satisfying the superradiant condition, $0 < \hat{\omega} < m\Omega$ will **grow**, $\nu > 0$. The co-rotating dipole mode $l = m = 1$ is dominant

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- For a pion π^0 + astrophysical BH, $M\mu \sim 10^{18}$ (!)
- The instability is only significant for primordial black holes ... or **ultra-light bosonic fields** such as axions.

Growth of bound states: Key results

- Zouros & Eardley (1979):

$$M\nu \sim 10^{-7} e^{-1.84 M\mu}, \qquad M\mu \gg 1.$$

- Detweiler (1980):

$$M\nu \sim -\frac{1}{12}(M\mu)^9 \, (\mu - \Omega) \, r_+, \qquad M\mu \ll 1, \; l = 1$$

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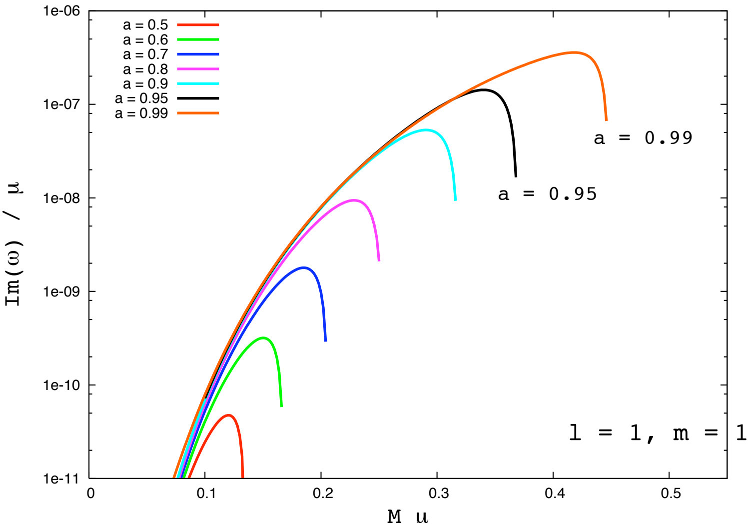
- Numerical results for intermediate regime $M\mu \sim 1$ found by Furuhashi *et al.* (2004), Cardoso *et al.* (2005), me (2007) and others.

- Minimum e-folding time $\tau_{\text{min}} = 1/\nu_{\text{max}}$,

$$\tau_{\text{min}} \approx 5.81 \times 10^6 \, GM/c^3 \approx 29 \text{ sec} \times \left(\frac{M}{M_\odot}\right)$$

for $a \approx 0.997M$ and $M\mu \approx 0.45$.

Unstable Bound States: $M\nu > 0$



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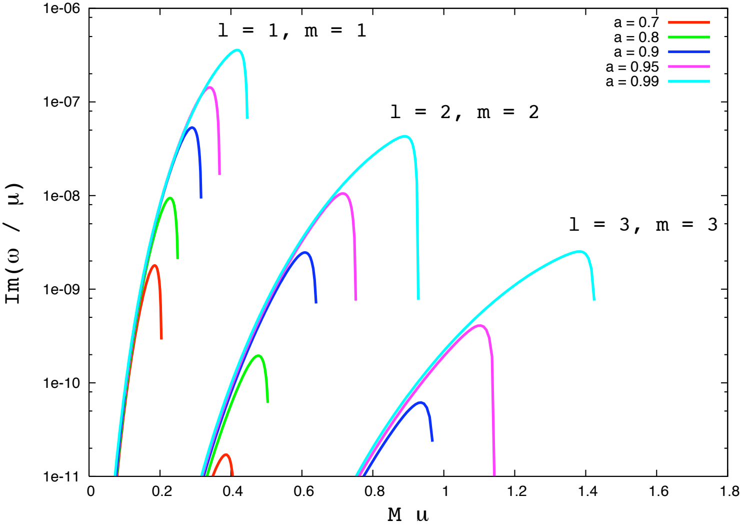
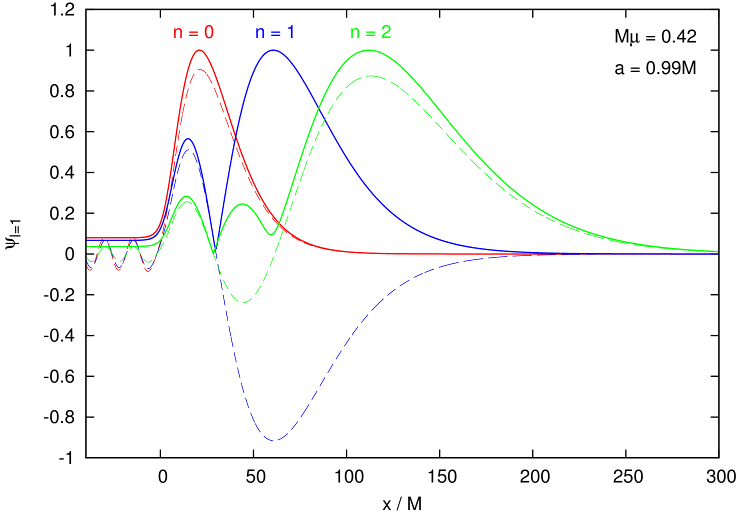
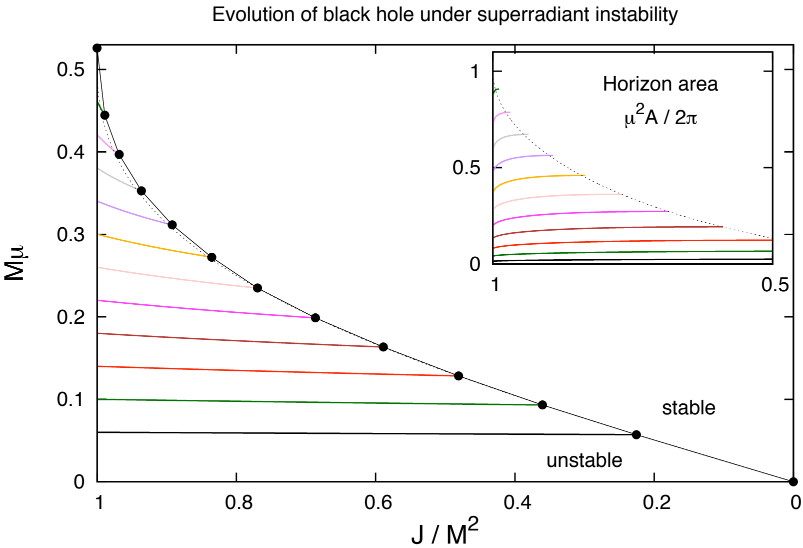


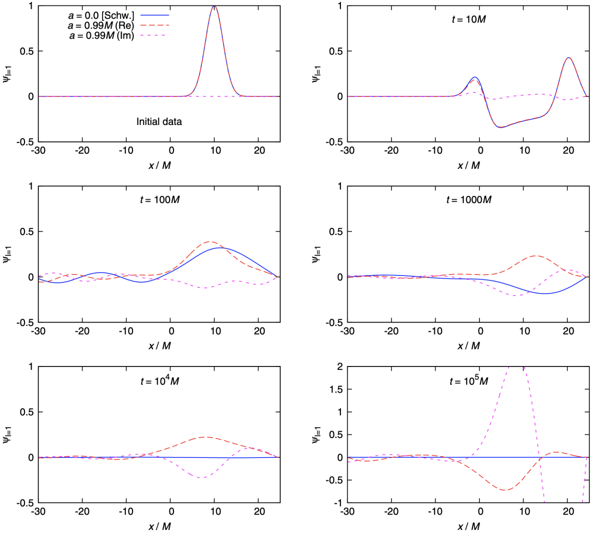
What is the profile of the bound states?



How do BH parameters evolve?



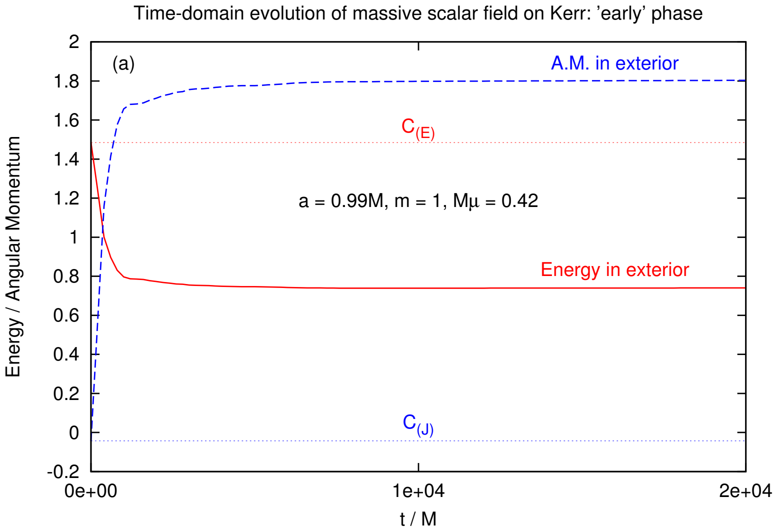
Evolution of massless scalar field on Kerr spacetime



$\mu = 0$, mirror at $r = 20M$.

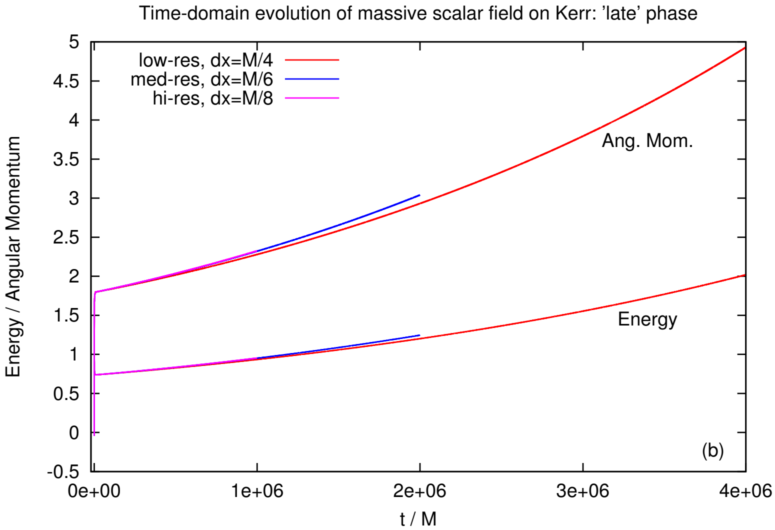
Evolution of massive scalar field on Kerr spacetime

'Early' times: $t \lesssim 10^4 M$



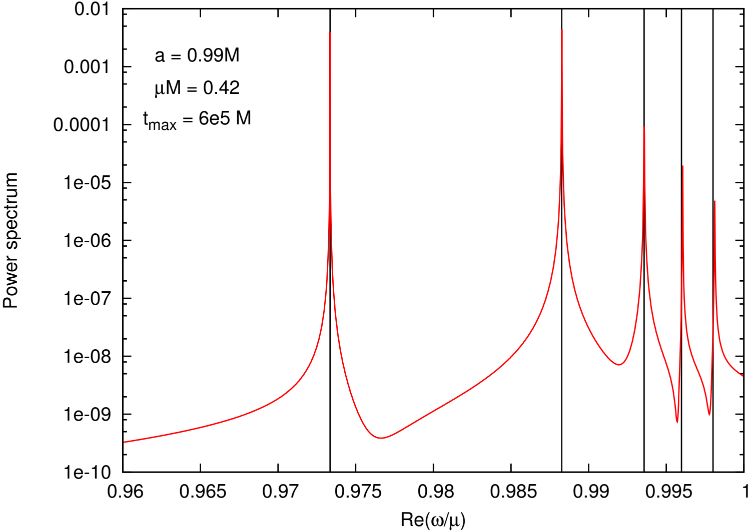
Evolution of massive scalar field on Kerr spacetime

'Late' times: $t \lesssim 4 \times 10^6 M$



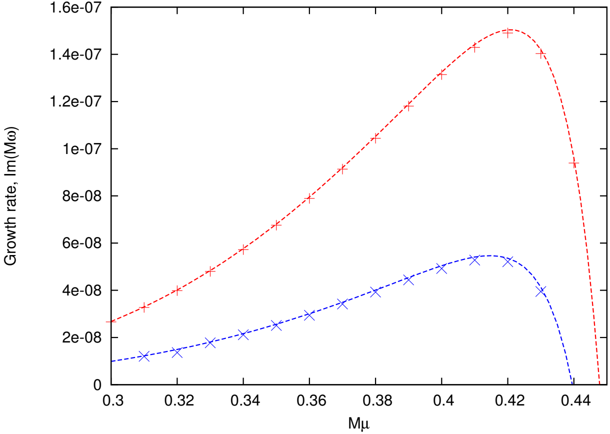
Evolution of massive scalar field on Kerr spacetime

Fourier analysis: recovering the bound state spectrum



Evolution of massive scalar field

Fourier analysis: recovering the growth rate of $n = 0$ and $n = 1$



Lines: frequency-domain results.

Points: growth rates extracted from time domain runs.

4. Massive (dark) photons and gravitons?

Massive spin-one fields

- What if a spin-1 field had a mass $\mathfrak{m}$?

$$\nabla_\beta F^{\alpha\beta} + \mathfrak{m}^2 A^\alpha = 0.$$

- 'The string photiverse': non-trivial gauge field configurations.

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- 'The string photiverse': non-trivial gauge field configurations.
- No gauge freedom: $\mathfrak{m} \neq 0 \Rightarrow \nabla_\alpha A^\alpha = 0$.
- Three physical polarizations $S = -1, 0, +1$.
- Under spatial inversion, $S = +1$ and $S = -1$ are even-parity, and $S = 0$ is odd-parity.

Bound states of the Proca field

Calculating the growth rate of the Proca field was a challenge of interest for several years. Highlights include:

- *"Superradiant instabilities in astrophysical systems"*, Witek, Cardoso, Ishibashi & Sperhake, Phys. Rev. D **87**, 043513 (2013).
- *"Black-Hole Bombs and Photon-Mass Bounds"*, Pani, Cardoso, Gualtieri, Berti & Ishibashi Phys. Rev. Lett. **109**, 131102 (2012).
- *"Superradiant Instability and Back-reaction of Massive Vector Fields around Kerr Black Holes"*, East & Pretorius, Phys. Rev. Lett. **119**, 041101 (2017).
- *"A modern approach to superradiance"*, Endlich & Penco, JHEP 2017: 52 (2017).
- *"Black Hole Superradiance Signatures of Ultralight Vectors"*, Baryakhtar, Lasenby & Teo, Phys. Rev. D **96**, 035019 (2017).
- *"Massive Vector Fields in Kerr-NUT-(A)dS Spacetimes: Separability and Quasinormal Modes"*, Frolov, Krtous, Kubiznak & Santos, Phys.Rev.Lett. **120**, 231103 (2018).

Bound states of the Proca field

- Baryakhtar, Lasenby & Teo (2017) derived an analytic approximation for the growth rate:
$$\mathrm{Im}(\omega) \sim (M\mu)^{2j+2l+5}(m\Omega - \omega)$$

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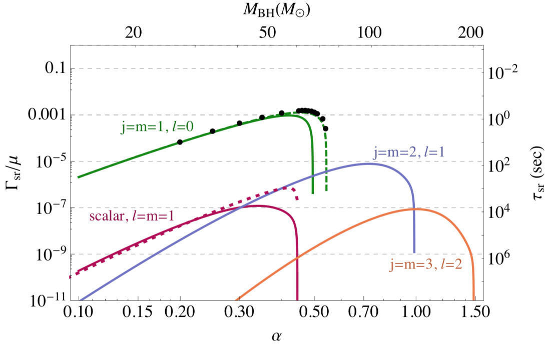
- East (2017) obtained numerical data for the growth rate from time-domain simulations. This is Fig. 2 from BLT ↓



Separability of the Proca field

- In 2018, Frolov, Krtous, Kubiznak & Santos showed that the equations governing the even-parity ($S = \pm 1$) modes of the Proca field are separable!

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- With the ansatz $A^\mu = B^{\mu\nu}\nabla_\nu \Psi$ for the vector field, and a multiplicative separability ansatz for Ψ, FKKS found that

$$\begin{aligned} \frac{d}{dr}\left[\Delta \frac{dR}{dr}\right] + \left[\frac{K_r^2}{\Delta} + \frac{2-q_r}{q_r}\frac{\sigma}{\nu} - \frac{q_r \mu^2}{\nu^2}\right] R(r) &= 0 \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dS}{d\theta}\right] - \left[\frac{K_\theta^2}{\sin^2\theta} + \frac{2-q_\theta}{q_\theta}\frac{\sigma}{\nu} - \frac{q_\theta \mu^2}{\nu^2}\right] S(\theta) &= 0 \end{aligned}$$

where

$$\begin{aligned} K_r &= am - (a^2 + r^2)\omega, \qquad K_\theta = m - a\omega \sin^2\theta, \\ q_r &= 1 + \nu^2 r^2, \qquad q_\theta = 1 - \nu^2 a^2 \cos^2\theta, \qquad \sigma = \omega + a\nu^2(m - a\omega). \end{aligned}$$

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- Here ν is the separation constant (impose regularity on $S(\theta)$ at poles).
- In the limit $a \to 0$, $S = Y_{lm}(\theta)$ and $\omega/\nu - \mu^2/\nu^2 = -l(l+1)$.

Separability of the Proca field

- The tensor $B^{\mu\nu}$ in the ansatz $A^{\mu} = B^{\mu\nu}\nabla_{\nu}\Phi$ is related to the **principal tensor** $h_{\mu\nu}$ via
$$B^{\mu\nu}\left(g_{\nu\sigma} + i\nu h_{\nu\sigma}\right) = \delta^{\mu}_{\sigma}$$
- For technical details see
  - *"Massive Vector Fields in Kerr-NUT-(A)dS Spacetimes: Separability and Quasinormal Modes"*, FKKS, arXiv:1804.00030.
  - *"Separation of Maxwell equations in Kerr-NUT-(A)dS spacetimes"*, KFK, arXiv:1803.02485.
  - *"Black holes, hidden symmetries, and complete integrability"*, FKK, Living Rev. Relativ. (2017) 20:6; arXiv 1705.5482.
- The decoupled ODEs can be solved numerically in the usual way (direct integration) to find the bound state spectrum.

Proca field: results

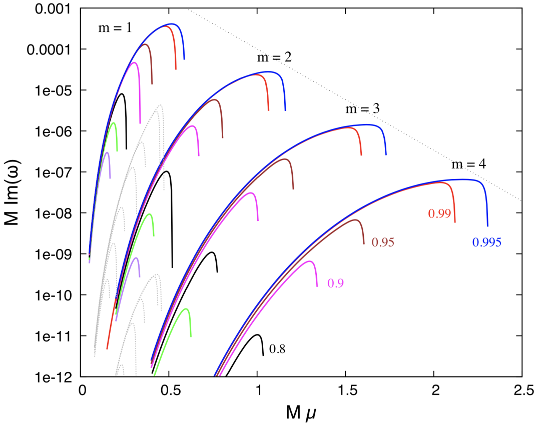

FIG. 5. Instability growth rates for $m = 1, 2, 3$ and 4. The solid lines show the $S = -1$ modes for $a \in \{0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.995\}$. The dotted lines show the $S = 0$ and $S = -1$ modes for $m = 1$.

SRD, Phys. Rev. D **98**, 104006 (2018).

Proca field: comparison

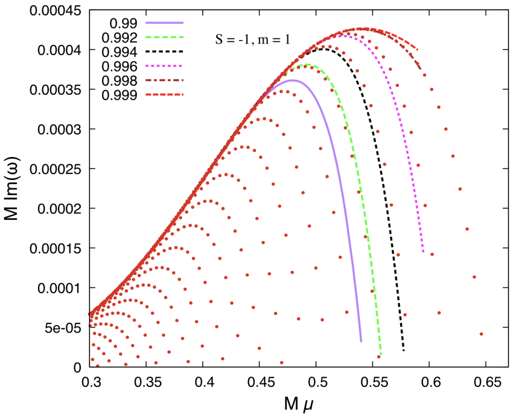

FIG. 6. The maximum growth rate for the dominant $S = -1$, $m = 1$ mode. The lines show new data obtained here by solving the ODEs. The points show the data set[2] that Cardoso *et al.* [23] obtained by solving PDEs.

PRD **98**, 104006 (2018) vs V. Cardoso et al, Astropart. Phys. 03 (2018) 043.

Massive spin-two field?

$$
\begin{aligned}
&\Box H_{ab} + 2R_{abcd}H^{cd} - \mu^2 H_{ab} = 0\,, \\
&\nabla^a H_{ab} = 0\,, \qquad H^a{}_a = 0\,,
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Dias et al, Phys. Rev. D **108** (2023) 4, L041502.

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- A special **dipole** mode dominates, $\omega_i \propto (M\mu)^3(\omega_r - m\Omega)$.

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- For nearly-extremal Kerr BH, $M\omega_i \approx 0.019$ for $M\mu = 0.8$, and

$$\tau \approx 2.6 \times 10^{-4} \left(\frac{M}{M_\odot}\right) \text{sec.}$$

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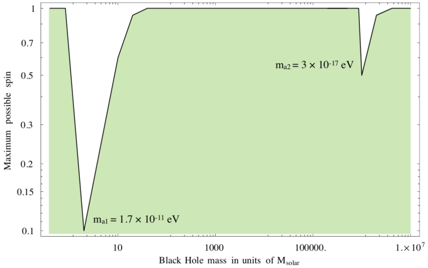
$$\tau \approx 2.6 \times 10^{-4} \left(\frac{M}{M_\odot}\right) \text{sec.}$$

- Almost two orders of magnitude **shorter** than any other superradiant mode.

5. Black holes as particle detectors

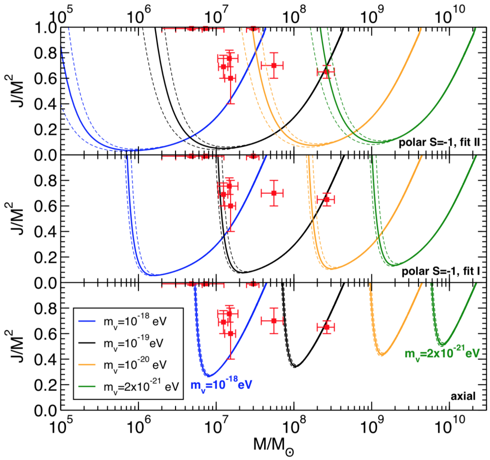
Mass gaps in the Regge plane

- Arvanitaki-Dimopoulos-Dubovsky *et al* '09: If accretion is not efficient enough to support the superradiant instability, then the black hole will spin down. Since the instability rate is highly sensitive to $M\mu$, this would lead to "gaps" appearing in the Regge plane which diagnose the mass of the axion / ultra-light field.



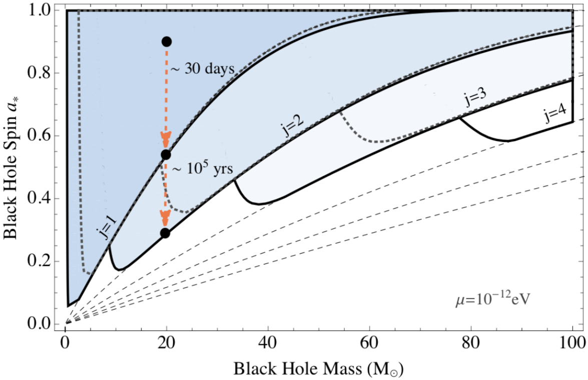
"The maximum allowed spin for a black hole as a function of its mass assuming there are two axions with mass m_{a1} and m_{a2} corresponding roughly to black hole masses of $2M_\odot$ and $10^6 M_\odot$."

Mass gaps in the Regge plane: Proca field



- Fig. 1 in Pani *et al.* (2012), “Perturbations of slowly-rotating ...”
- Red crosses = survey data.

Mass gaps in the Regge plane: Scalar & Proca



- Fig. 1 in BLT (2017) "Black Hole Superradiance Signatures ..."
- Solid = Proca. Dotted = Scalar field.

Hairy black holes

- Various theorems show that 4D black holes with hair **cannot exist** ...
- ... **but** the theorems typically use an axiom that the field shares two Killing symmetries $\xi^a_{(t)} \nabla_a \Phi = 0 = \xi^a_{(\phi)} \nabla_a \Phi$.
- In 2014, Herdeiro & Radu evaded this axiom by considering a **helically-symmetric complex field** $(\xi^a_{(t)} + \Omega \xi^a_{(\phi)}) \nabla_a \Phi = 0$ generating a **stationary** stress-energy $T_{ab} = 0$.
- They identified a class of 4D scalar-hairy (and Proca-hairy) black holes connecting the Kerr black hole to the boson star family.
- Work by East & Pretorius suggests that the superradiant instability can evolve towards to a Herdeiro-Radu hairy black hole.

Hairy black holes: phase diagram

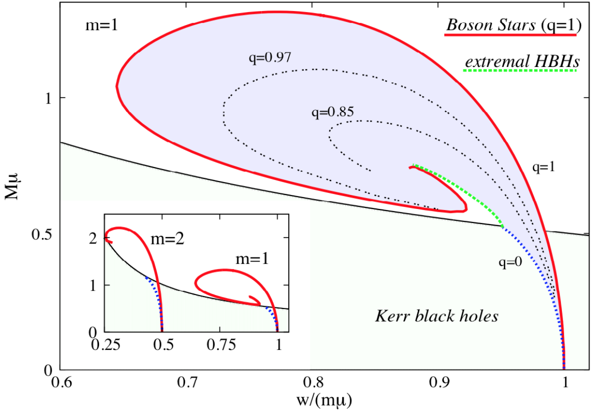


Fig. 2 from "Kerr black holes with scalar hair", PRL, Herdeiro & Radu (2014).

We will learn more about possible astrophysical signatures of superradiant instabilities in the talks this week.

Thank you for listening.